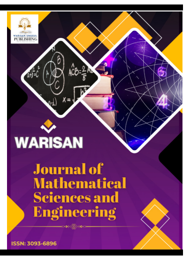
WARISAN UNGGUL PUBLISHING

Warisan Journal of Mathematical Sciences and Engineering

Journal homepage:
https://warisanunggul.my/index.php/wjmse/index
ISSN: 3093-6896



Comparative Study of the Differential Transform Method and the α-Parametrized Differential Transform Method for Boundary Value Problems

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ARTICLE INFO

Article history:
Received 8 August 2025
Received in revised form 24 August 2025
Accepted 27 August 2025
Available online 13 September 2025

Keywords:
Boundary value problems; differential transform method; α-parametrized differential transform method

ABSTRACT

In this paper, the differential transform method (DTM) and the α-parametrized differential transform method (α–PDTM) are studied and linked by applying them to boundary value problems. Numerical solution obtained from the examples were compared with that of exact solution. The absolute error obtained from this evaluation showed that DTM has lower error margin and rapid convergence when likened to α–PDTM. Tables for discrete and exact solutions were presented to show the efficiency of the two methods. Numerical computations were added and solution obtained showed that DTM is very precise and proficient method in distinction to α–PDTM.

1. Introduction

Boundary value problems (BVPs) are used to model a variety of physical phenomena in all fields of natural research. Different methods of analysis are used to discover the accurate solutions of a particular type of linear differential equation. However, not all differential equations can be solved methodically. In general, basic analytical methods are incapable to fully handle non-linear and distinctive boundary value problems. Consequently, a range of approximatively and numerical techniques, such as the Variational Iterative Method (VIM), Adomian Decomposition Method (ADM), Finite Difference Method (FDM), Differential Transform Method (DTM), Homotopy Perturbation Method (HPM), and others, are helpful tools for recognizing and scrutinizing through the qualitative appearances of various differential equations with precise unknown solutions.

The differential transform method (DTM) is one of the most popular and useful algorithms among semi-analytical techniques. To address IVPs in the field of electrical circuits, Zhou [1] originally

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https://doi.org/10.37934/wjmse.2.1.110

introduced this method in 1986.The technique's foundation comes from calculating the coefficients of the problem's solution's Taylor series. The technique was created to solve calculus of variations, integral equations, BVPs in one or more dimensions, and optimal control. It has been applied to the analysis of various physical phenomena with fractional and stochastic behavior, particularly in the last ten years.

The fundamental benefit of this approach is that it doesn't require linearization and may be used directly on nonlinear ODEs. The well-known benefit of DTM is its broad variety of applications, extensive calculation accuracy, and ease of use. Another significant benefit is that this approach can significantly minimize the amount of computational work required while still accurately and quickly producing the series solution. This approach can be used to get exact solutions to differential equations or highly accurate findings.

Chiou and Tzeng [2] utilized the Taylor transform to deal with nonlinear vibration problems, Chen, Ho [3] established this approach to solve a variety of linear and nonlinear issues, including boundary value issues with two points, and Ayaz [4] performed it on the system of differential equations. Abbasov *et al*., [5] used the method of differential transform to obtain approximate solutions of the linear and non-linear equations related to engineering problems and observed that the numerical results are in good agreement with the analytical solutions. In recent years, many authors have used this method for solving various types of equations. For example, this method has been used for differential-algebraic equations [6], partial differential equations [3,7-9], fractional differential equations [10] and difference equations [11]. Abazari *et al*., [12] applied this method for Schr¨odinger equations [12]. Different applications of DTM can be found in [13,14]. Khudair *et al*., [15] investigated some second-order random differential equations by DTM. Ünal and Gökdoğan [16] generalized DTM to solve not only classical differential equations but also fractional differential equations. Biswas and Roy [17] devoted the intuitionistic (fuzzy) differential transform method to solving Volterra-type fuzzy integrodifferential equations. Mukhtarov *et al*., [18] suggested a new generalization of DTM to investigate some spectral properties of a new type of boundary-value problem. The differential transform method (DTM), while a powerful numerical technique for resolving numerous initial value issues, is not without disadvantages. This is because the DTM is designed for problems with analytic solutions, or answers that can be extended into Taylor series. Our aim in this study is to tackle boundary value problems using DTM together with a modified version we call the α-parametrized differential transform method (α–PDTM).

2. Materials and Methods

2.1 Differential Transform Method (DTM):

The DTM is developed based on Taylor series expansion and constructs an analytical solution in the form of polynomial.

2.1.1 Definition 1:

Taylor series of real or complex-valued function *f(y)* that is infinitely differentiable at real or complex number *a* is a power series.

$$f(y) = f(a) - \frac{f\prime(a)}{1!}(x-a) + \frac{f\prime\prime(a)}{2!}(x-a)^2 + \cdots \quad (1)$$

For n degree:

$$f_n(y) = \sum_{k=0}^{n} \frac{1}{k!}(f^k(a))(y-a)^k \quad (2)$$

$$f_n(y) = \sum_{k=0}^{n} \frac{1}{k!}(x^k(a))(y-a)^k \tag{3}$$

Consider a function *y(x)* which is analytical in domain *D* and let $x = x_0$ represent any point in *D*. The function *y(x)* is then represented by a power series whose center is located at x_0.The differential transformation of the function *y(x)* is given by

$$Y(K) = \frac{1}{K!}\left(\frac{d^k y(x)}{dx^k}\right)_{x=x_0} \tag{4}$$

Where *y(x)* is original function, *Y[K]* is transformed function.

Then, the inverse transformation is defined as,

$$y(x) = \sum_{k=0}^{\infty}(x-x_0)^k Y(k) \tag{5}$$

By combining equation (2.4) and (2.5)

$$y(x) = \sum_{k=0}^{\infty}\frac{(x-x_0)^k}{k!}\left(\frac{d^k y(x)}{dx^k}\right)_{x=x_0} \tag{6}$$

The fundamental mathematical operations performed by the differential transform method are listed in following

i. $y(x) = r(x) \pm p(x)$ then $Y(k) = R(k) \pm P(k)$
ii. $y(x) = \alpha r(x)$ then $Y(k) = \alpha R(k)$
iii. $y(x) = \frac{dr(x)}{dx}$ then $Y(k) = (k+1)R(k+1)$
iv. $y(x) = \frac{d^2r(x)}{dx^2}$ then $Y(k) = (k+2)R(k+2)$
v. $y(x) = \frac{d^br(x)}{dx^b}$ then $Y(k) = (k+1)(k+2)\dots(k+b)R(k+b)$
vi. $f(x) = g(x)h(x)$ then $F(k) = \sum_{r=0}^{k} G(r)H(k-r)$

2.2. α-Parameterized Differential Transform Method (α-pDTM):

In this section, we suggest a new version of classical differential transform method by following.

2.2.1 Definition 2:

Let *g*= [*b, c*] ⊂*R* be arbitrary real interval, *h: g→R* is function with infinite differentiation, α∈ [0, 1] any real parameter and *N* any integer,
Then following notation represent it as,

$$D_\alpha(h, \alpha; k) = \alpha D_b(h; k) + (1-\alpha)D_c(h; k) \tag{7}$$

Where $D_b(h;k) = \frac{h^k(b)}{k!}$, $D_c(h;k) = \frac{h^k(c)}{k!}$, $\alpha \in [0,1], k \in N$ (8)

2.2.2 Definition 3:

The sequence

$$D_\alpha(h) = (D_\alpha(h,\alpha;1), D_\alpha(h,\alpha;2), \dots) \tag{9}$$

Is known as the α-P transformation of the original function *h(x).*

The differential inverse transform of $D_\alpha(h)$ is defined as

$$E_\alpha\big(D_\alpha(h)\big) = \sum_{k=0}^{\infty} D_\alpha(h,\alpha;k)(x - x_\alpha) \tag{10}$$

Should the series converge, where $x_\alpha = \alpha b - (1-\alpha)c$.

If α=1 and α=0, then it will reduce to classical DTM at point *x=b* and *x=c*.

The function $\tilde{h}_\alpha(x)$ defined as

$$\widetilde{h_\alpha}(x) = E_\alpha(D_\alpha(h)) \tag{11}$$

Is said to be the α-parameterized approximation of the function *h(x).*

2.2.3 Definition 4:

An Nth α-parameterized approximation of the function $\tilde{h}_\alpha(x)$ is explored as

$$\widetilde{h_{\alpha,N}}(x) = E_{\alpha,N}\big(D_\alpha(h)\big) = \sum_{k=0}^{N} D(h,\alpha;k)(x - x_\alpha) \tag{12}$$

By using Definition 2, we can show that α- parameterized differential transform has the following properties:

i. $h(x) = kg(x), k \in R$, Then $D_\alpha(h) = kD_\alpha(g)$.
ii. $h(x) = p(x) \pm q(x)$ Then $D_\alpha(p) \pm D_\alpha(q)$.
iii. $h(x) = \frac{d^a g}{dx^a}, a \in N$ Then $D(h^a,\alpha;k) = \frac{(k+a)!}{k!} D(h,\alpha;k+a)$.

3. Results and Discussion

3.1. Example 1

Consider the Boundary value problem

$$y''(x) + y(x) = \cos(x), \quad y(0) = 1, y(1) = 0 \tag{13}$$

Exact solution for equation (13)

$$y(x) = \frac{e^{-x}}{2(-1+e^2)}(3e^2 - 3e^{2x} + 2e^{1+2x} - e\cos(1) + e^{1+2x}\cos(1) + e^x\cos(x) - e^{2+x}\cos(x) \dots \tag{14}$$

3.1.1 Differential transform of example 1

After applying DTM on equation (13)

$(k+1)(k+2)Y(k+2) = Y(k) + \frac{1}{k!}cos\left(\frac{k\pi}{2}\right)$ (15)

$Y(k+2) = \frac{1}{(k+1)(k+2)}\left[Y(k) + \frac{1}{k!}cos\left(\frac{k\pi}{2}\right)\right]$ (16)

Let$Y(0) = A\ and\ Y(1) = B$, we have for $Y(k)$

$Y(2) = \frac{1+A}{2}, Y(3) = \frac{B}{6}, Y(4) = \frac{A}{24}, Y(5) = \frac{B}{120}, Y(6) = \frac{1+A}{720}, Y(7) = \frac{B}{5040}, \ldots$ (17)

Series solution of DTM

$y(x) = \sum_{k=0}^{\infty} Y(k)x^k = A + Bx + \frac{1+A}{2}x^2 + \frac{B}{6}x^3 + \frac{A}{24}x^4 + \frac{B}{120}x^5 + \cdots$ (18)

By applying boundary condition, we get our series expansion

$y(x) = 1.499577x + 9971836x^3 + 0.1999436x^5 + 0.019042247\, x^7 + \cdots$ (19)

3.1.2 α– Parametrized Differential transform of example 1

After applying $\alpha - PDT$M on equation (13)

$(k+1)(k+2)D(y,\alpha;k+2) - D(y,\alpha;k) = \frac{1}{k!}cos\left(\frac{k\pi}{2}\right)$ (20)

$D(y,\alpha;k+2) = \frac{1}{(k+1)(k+2)}\left[D(y,\alpha;k) + \frac{1}{k!}cos\left(\frac{k\pi}{2}\right)\right]$ (21)

Let $D(y,\alpha;0) = A$, $D(y,\alpha;1) = B$

Put $k = 0,1,2,3,4,5,6,7 \ldots$, we calculate

$D(y,\alpha;2) = \frac{1+A}{2}, D(y,\alpha;3) = \frac{B}{6}, D(y,\alpha;4) = \frac{2A}{3}, D(y,\alpha;5) = \frac{2B}{15}, D(y,\alpha;6) = \frac{4A}{45}, D(y,\alpha;7) = \frac{4B}{315}$ (22)

By definition of $\alpha - PDT$M

$$y_\alpha(x) = \sum_{k=0}^{\infty} D(y,\alpha;k)(x - x_\alpha)^k \quad (23)$$

Where $x_\alpha = \alpha a + (1-\alpha)b$

We get

$y_\alpha(x) = \sum_{k=0}^{\infty} D(y,\alpha;k)(x - 1 + \alpha)^k$ (24)

By applying boundary conditions

$$1 = \sum_{k=0}^{\infty} D(y,\alpha;k)(\alpha - 1)^k \quad (25)$$

$$0 = \sum_{k=0}^{\infty} D(y,\alpha;k)(\alpha)^k \quad (26)$$

By utilizing equation (24), (25) and (26) with $\alpha = \frac{1}{20}$, we get

$$A = 0.95779, \quad B = 0.79479 \tag{27}$$

Hence α – parametrized series solution is obtained up to *N=7*

$$y_\alpha(x) = \sum_{k=0}^{\infty} D(y, \alpha; k)(x - 1 + \alpha)^k \tag{28}$$

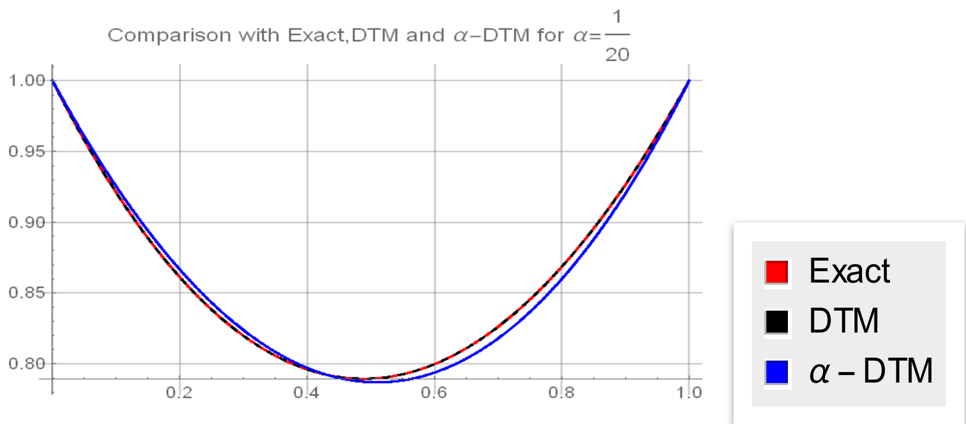
$$= 0.95779 + 0.79479(x + \alpha - 1) + 0.97889(x + \alpha - 1)^2 + 0.13246(x + \alpha - 1)^3 + 0.039907(x + \alpha - 1)^4 + 0.0066232(x + \alpha - 1)^5 + 0.0027191(x + \alpha - 1)^6 + 0.00015769(x + \alpha - 1)^7 \ldots \tag{29}$$

Table for example 1

Table 1
Error obtained for example 1 using DTM and α-pDTM with k = 0...7

X	EXACT	DTM	α-pDTM $\alpha = 1/20$	ERROR WITH DTM	ERROR WITH α-pDTM
0.0	1.000000	1.000000	1.001888	0.000000	-0.001888
0.2	0.861128	0.861127	0.867959	0.000001	-0.006831
0.4	0.796019	0.796006	0.798123	0.000013	-0.002104
0.6	0.799699	0.799571	0.794417	0.000128	0.005282
0.8	0.868488	0.867789	0.860169	0.000699	0.008319
1.0	1.000000	0.997371	0.999994	0.002629	0.000006

Graph for example 1



3.2 Example 2

Consider the Boundary value problem

$$y''(x) - 4y(x) = \sin(2x)\,, y(0) = 1, y'(1) = 0 \tag{30}$$

Exact solution for equation (30)

$$y(x) = \frac{1}{8(1+e^4)} e^{-2x}(8e^4 + 8e^{4x} - e^2\mathrm{Cos}[2] + e^{2+4x}\mathrm{Cos}[2] - e^{2x}\mathrm{Sin}[2x] - e^{4+2x}\mathrm{Sin}[2x])\ldots \tag{31}$$

3.2.1 Differential transform of example 02:

After applying DTM on equation (30)

$$(k+1)(k+2)Y(k+2) = 4Y(k) + \frac{2^k}{k!}sin\left(\frac{k\pi}{2}\right) \quad (32)$$

$$Y(k+2) = \frac{1}{(k+1)(k+2)}[4Y(k) + \frac{2^k}{k!}sin\left(\frac{k\pi}{2}\right)] \quad (33)$$

Let $Y(0) = A \ and \ Y'(1) = B$, we have for $Y(k)$

$$Y(2) = 2A, Y(3) = \frac{(1+2B)}{3}, Y(4) = \frac{2A}{3}, Y(5) = \frac{2B}{15}, Y(6) = \frac{4A}{45}, Y(7) = \frac{(1+2B)}{315}, \dots \quad (34)$$

Series solution of DTM

$$y(x) = \sum_{k=0}^{\infty} Y(k)x^k = A + Bx + 2Ax^2 + \frac{1}{3}(1+2B)x^3 + \frac{2Ax^4}{3} + \frac{2B}{15}x^5 + \frac{1+2B}{315}x^6 + \cdots \quad (35)$$

By applying boundary condition, we get our series expansion

$$y(x) = 1 - 2.19527x + 2x^2 - 1.13018x^3 + \frac{2x^4}{3} - 0.292702x^5 + \frac{4x^6}{45} - 0.0215272\, x^7 + \cdots \quad (36)$$

3.2.1 α – Parametrized Differential transform of example 02:

After applying $\alpha - PDT$M on equation (30)

$$(k+1)(k+2)D(y,\alpha;k+2) - 4D(y,\alpha;k) = \frac{2^k}{k!}sin\left(\frac{k\pi}{2}\right) \quad (37)$$

$$D(y,\alpha;k+2) = \frac{1}{(k+1)(k+2)}\left[4D(y,\alpha;k) + \frac{2^k}{k!}sin\left(\frac{k\pi}{2}\right)\right] \quad (38)$$

Let $D(y,\alpha;0) = A$, $D(y,\alpha;1) = B$

Put $k = 0,1,2,3,4,5,6,7 \dots$, we calculate

$$D(y,\alpha;2) = 2A, \quad D(y,\alpha;3) = \frac{(1+2B)}{3}, D(y,\alpha;4) = \frac{2A}{3}, D(y,\alpha;5) = \frac{2B}{15}, D(y,\alpha;6) = \frac{4A}{45}, D(y,\alpha;7) = \frac{2(1+2B)}{315}, \dots \quad (39)$$

By definition of $\alpha - PDT$M

$$y_\alpha(x) = \sum_{k=0}^{\infty} D(y,\alpha;k)(x - x_\alpha)^k \quad (40)$$

Where $x_\alpha = \alpha a + (1-\alpha)b$

We get

$$y_\alpha(x) = \sum_{k=0}^{\infty} D(y,\alpha;k)(x - 1 + \alpha)^k \quad (41)$$

By applying boundary conditions

$$1 = \sum_{k=0}^{\infty} D(y,\alpha;k)(\alpha - 1)^k \quad (42)$$

$$0 = \sum_{k=0}^{\infty} D(y,\alpha;k)k(\alpha)^{k-1} \quad (43)$$

By utilizing equation (41), (42) and (43) with$\alpha = \frac{1}{20}$, we get

$$A = 0.343987, B = -0.0710565 \quad (44)$$

Hence α – parametrized series solution is obtained up to *N=7*

$$y_\alpha(x) = \sum_{k=0}^{7} D(y, \alpha; k)(x - 1 + \alpha)^k \quad (45)$$

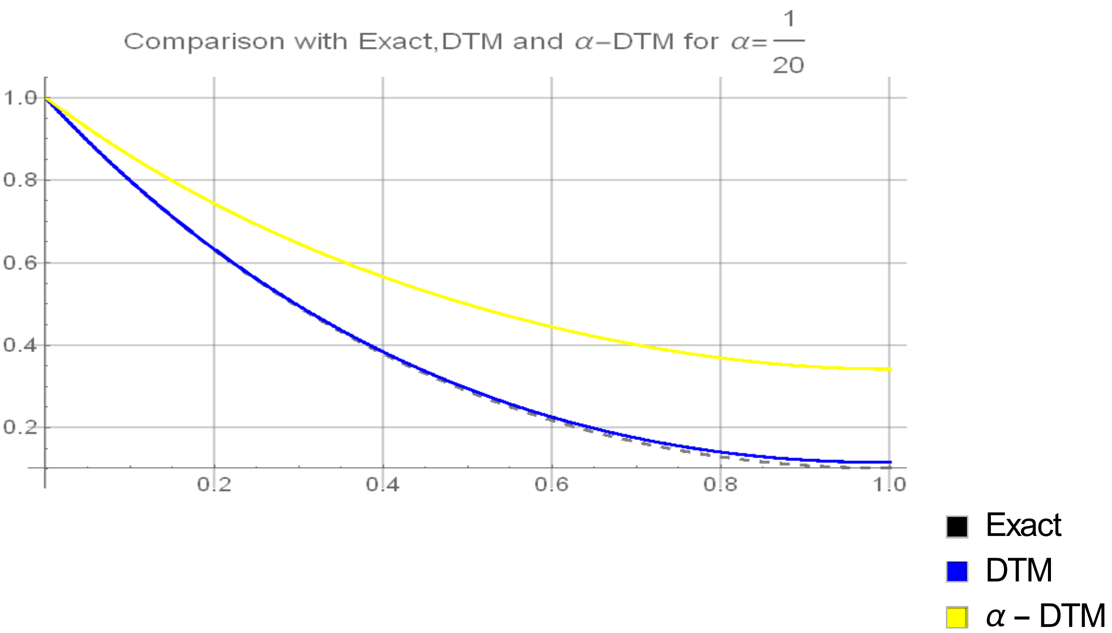
$$= 0.3439868651034321 - 0.07105651459417159(-\frac{19}{20} + x) + 0.6879737302068643(-\frac{19}{20} + x)^2 + 0.28596232360388557(-\frac{19}{20} + x)^3 + 0.2293245767356214(-\frac{19}{20} + x)^4 - 0.009474201945889549(-\frac{19}{20} + x)^5 + 0.030576610231416185(-\frac{19}{20} + x)^6 + 0.005446901401978773(-\frac{19}{20} + x)^7 \ldots \quad (46)$$

Table for example 02:

Table 2
Error obtained for Example 2 using DTM and α-pDTM with k = 0...7

X	EXACT	DTM	α-pDTM $\alpha = 1/20$	ERORR WITH DTM	ERORR WITH α-pDTM
0.0	1.000000	1.000000	1.000000	0.000000	0.000000
0.2	0.630739	0.632884	0.743147	-0.002145	-0.112408
0.4	0.379327	0.38396	0.565828	-0.004633	-0.186501
0.6	0.218118	0.225906	0.444417	-0.007788	-0.226299
0.8	0.129559	0.141077	0.369277	-0.011518	-0.239718
1.0	0.101993	0.115882	0.342191	-0.013889	-0.240198

Graph for example 02:



4. Conclusions

In this study, we effectively employ the DTM to get numerical solutions for both homogenous and non-homogenious ordinary differential equation with boundary and initial condition. The DTM has been shown to be a dependable and efficient method for solving systems of ordinary differential equations. Rapid convergence of series solutions is obtained by this approach. By include more terms in the solution; the accuracy of the final result can be increased. The DTM-obtained series solutions may often be expressed in precise closed form. The current approach simplifies all computations and eliminates the computing challenges of the other conventional approaches. Using the DTM, a number of cases were examined, and the outcomes demonstrated exceptional performance.

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