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Type-2 Fuzzy Interpolation Bezier Curve Model for Earthquake Magnitude Uncertainty Data Modeling

Rozaimi Zakaria[1,*], Wilvaldo Dousip[1], Rodeano Roslee[2,] Suzelawati Zenian[1], Mohammad Izat Emir Zulkifly[3]

[1] Mathematics Visualization (MathViz) Research Group, Faculty of Science and Technology, Universiti Malaysia Sabah, Jalan UMS, 88400 Kota Kinabalu, Sabah, Malaysia
[2] Faculty of Science and Technology, Universiti Malaysia Sabah, Jalan UMS, 88400 Kota Kinabalu, Sabah, Malaysia
[3] Department of Mathematics, Faculty of Sciences, University of Technology Malaysia (UTM), Johor Bahru, Malaysia

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ABSTRACT

Uncertainty is a fundamental characteristic of real-world data, especially in the modeling of curves and surfaces, where variability can be intricate and challenging to accurately represent. Conventional modeling methods, dependent on static numerical representations, often struggle to deliver a dependable characterization of uncertainty, leading to diminished accuracy and restricted interpretability. This study examines the issue of insufficient representation of complex uncertainty in data modeling, emphasizing the necessity for methodologies that can incorporate variability while maintaining critical structural information. This research aims to present a more advanced framework for uncertainty modeling by utilizing type-2 fuzzy numbers (T2FN), which provide enhanced capabilities for characterizing imprecision in comparison to traditional methods. The proposed method utilizes a Type-2 fuzzy interpolation Bezier curve (T2FIBC), facilitating the visualization of uncertainty while effectively capturing the underlying data trends. The model leverages the descriptive capabilities of type-2 fuzzy sets, offering a more nuanced and adaptable representation of uncertainty. Experimental results indicate that the T2FIBC approach attains superior accuracy and robustness compared to conventional interpolation techniques, especially in situations characterized by considerable variability. The model further improves the visualization of uncertainty, providing more precise insights into the behavior of intricate datasets. In conclusion, this research validates the effectiveness of Type-2 fuzzy interpolation Bézier curves as a significant asset for addressing uncertainty in data modeling. This approach not only enhances the analysis of geoscientific datasets, including earthquake magnitude records, but also offers a versatile framework applicable to various fields where effective representation of uncertainty is crucial for reliable decision-making.

* *Corresponding author.*
E-mail address: rozaimi@ums.edu.my

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1. Introduction

Data modeling plays a crucial role in numerous disciplines, including science, engineering, and the social sciences. This enables researchers and practitioners to discern patterns, predict outcomes, and enhance planning and optimization initiatives. Meteorological forecasts, financial risk assessments, and disease modeling in healthcare depend heavily on the ability of data models to transform complex datasets into valuable insights that support informed decision-making.

While these models provide valuable insights, their reliability diminishes when faced with uncertainty arising from measurement errors, incomplete data, or environmental fluctuations. This challenge holds considerable importance in seismology, as earthquake magnitudes may vary depending on the measurement techniques utilized and the locations of observation. Conventional modeling approaches, reliant on precise and fixed numerical values, frequently struggle to accommodate uncertainty, thus limiting their ability to effectively depict real-world phenomena.

Fuzzy set theory, first proposed by Zadeh in 1965 [1], introduced a flexible framework for handling imprecision in data. Type-1 fuzzy sets [2] have since been widely adopted, particularly because they allow the representation of vague information through membership functions. However, these sets assume that membership grades are clearly defined, a condition that does not hold in many real-world contexts where higher levels of uncertainty are present.

To address this limitation, Type-2 fuzzy sets were introduced, expanding the concept of fuzziness to include the membership functions themselves. The Footprint of Uncertainty (FOU) concept enables Type-2 fuzzy sets to capture “uncertainty within uncertainty,” thereby offering a more thorough and robust framework for modeling ambiguous datasets where discussed by several authors [3–5]. This makes them especially important in fields that rely on specialized knowledge and incomplete information.

At the same time, interpolation techniques such as Bézier curves have seen widespread application in computer graphics and data approximation. Their ability to generate smooth curves has been incorporated into fuzzy modeling, leading to the development of fuzzy Bézier curves that encompass uncertain control points which had been discuss by the reserchers [6-9]. Recent advancements encompass the integration of T2FNs with Bézier interpolation, significantly improving the capacity of these models to represent complex uncertainty as discussed in [2-6].

Although significant progress has been made, the use of Type-2 fuzzy interpolation Bézier models in the analysis of seismic data has not been thoroughly investigated. Many previous studies depend on Type-1 fuzzy numbers or traditional interpolation techniques, which do not sufficiently account for the significant variability and uncertainty inherent in earthquake magnitude data. This gap highlights the necessity for a more thorough framework that can effectively integrate T2FNs with Bézier interpolation to better address uncertainty. This study focuses on the development of a Type-2 fuzzy interpolation Bézier curve model specifically designed for seismic datasets. This study holds considerable importance as it has the potential to enhance the accuracy and reliability of uncertainty modeling, thereby offering more dependable insights for seismology and other domains where decision-making relies on the effective management of complex uncertainty.

2. Methodology

This method utilizes T2FNs and interpolation Bézier curves to provide a novel approach for representing earthquake magnitude data. The first step entails the collection and refinement of seismic data to guarantee its comprehensiveness and precision. T2FNs are employed to define control points for managing the uncertainties related to earthquake measurements. A type-2 fuzzy

Bézier curve is defined through parametric representations. This curve illustrates adaptability and can be employed to express uncertainty in multiple ways. The evaluation of the curves in representing the complexities of earthquake magnitude data is conducted through a comparison with conventional interpolation methods. This framework aims to improve predictive modeling, helping individuals make informed decisions about the assessment and management of seismic hazards. The suggested solution combines geometric adaptability with efficient strategies for handling uncertainty, highlighting its relevance in practical situations that involve complex datasets.

2.1 Data Collection

Data on earthquake magnitudes were obtained from the Malaysian Meteorological Department (MetMalaysia). The data consists of historical records of earthquake events that have taken place in the Malaysian region, particularly in Lahad Datu, Sabah. The dataset encompasses seismic events recorded from January 2017 to December 2017, featuring a range of magnitudes and locations.

2.2 Type-2 Fuzzy Number in Defining Uncertainty Complex Data

T2FNs play a crucial role in effectively characterizing complex uncertainties in data. They serve as a significant enhancement to conventional type-1 fuzzy numbers (T1FNs) by offering an additional dimension of flexibility in addressing ambiguity and uncertainty. The concept originated with Zadeh's development of type-2 fuzzy set theory in 1975 [12]. The objective was to enhance the modeling of uncertain data encountered in real-world scenarios which had been mentioned by Zakaria *et al*., Zhao *et al*., and Wahab and Zakariah [13-15]. T1FNs assume that membership functions are well-defined, whereas T2FNs allow membership values to be represented as fuzzy sets. This indicates that they may incorporate a Footprint of Uncertainty (FOU) that illustrates the level of ambiguity present in the data where discussed by researchers [16-18]. T2FNs may exhibit predicates that are ambiguous and address the inherent fuzziness that can occur during data collection or analysis [6,19]. For instance, Zakaria *et al*., [9] demonstrate that conventional models struggle to accurately measure these complexities and propose the use of T2FNs to more effectively represent uncertain data [20,21].

Modeling with type-2 fuzzy structures includes both theoretical definitions and practical applications, such as curve modeling. The integration of T2FNs with B-spline and Bézier curves creates a robust framework for interpolation and data representation [22,23]. The curves enhance the modeling of uncertainty in data sets by integrating variations in control points, effectively capturing underlying complexities and delivering a more precise representation of data trends in comparison to traditional methods as mentioned by researches as [13,22]. Furthermore, recent studies highlight the effectiveness of T2FNs in various engineering applications, showcasing their advantages over T1FNs in managing uncertainties in fields such as time series analysis and sensor fusion [14,24].

The combination of T2FNs with modern modeling techniques enables researchers to create models of uncertain data that demonstrate improved accuracy and dependability. T2FNs provide further dimensions for analysis, improving our understanding of data structures and facilitating more informed decisions and predictions [18,25].

2.3 Piecewise Interpolation Bezier Curve

Bézier curve interpolation is commonly utilized in computer graphics to represent and approximate data sets effectively. The smooth curves can be easily adjusted to move across a set of points using this method. This approach involves numerous Bézier segments, each characterized by its distinct set of control points. It ensures that the edges of the segments are smooth. Piecewise interpolation Bézier curves are advantageous as they maintain the integrity of a shape while minimizing the use of polynomial functions. This renders them more straightforward to manage compared to curves of a higher degree based on the previous study in Fadhel *et al*., [26]. This enables users to create curves that align well with the data and select the appropriate form for their specific context. This is especially important for achieving the correct visual outcomes in various domains, such as computer-aided geometric design (CAGD), animation, and graphical rendering. It can also employ piecewise Bézier curves to approximate actual shapes and functions effectively. This enhances the overall quality and precision in applications that require accuracy and smooth transitions.

Utilizing piecewise Bézier curve interpolation to model earthquake magnitude data provides an effective method for analyzing and elucidating the variations in magnitudes over time. Earthquake magnitude data often includes inherent variations and uncertainties due to the complexity of seismic events. Researchers can utilize Bézier curves to develop regression models that demonstrate the growth in size over time, while also considering the variations present in the data. This tool is highly effective for analyzing patterns and forecasting potential future earthquakes by utilizing historical data. For example, piecewise interpolation is capable of addressing both extremely high and low earthquake magnitudes, as well as abrupt changes in these values. This enables a more accurate representation of seismic activity over time based on the discussion in Bashir *et al*., [27]. Additionally, these curves can be modified to meet specific continuity requirements, making them a valuable resource for illustrating the variations in seismic measurements over time. This is essential for developing models that help individuals prepare for disasters as mentioned by Hanks *et al*., [28]. Utilizing piecewise Bézier curve interpolation to model earthquake magnitude data provides valuable insights into the data's significance and implications.

3. Result

The application of the proposed Type-2 fuzzy interpolation Bézier curve (T2FIBC) model produced significant improvements in modeling uncertainty in earthquake magnitude data. By integrating T2FNs with Bézier curve interpolation, the resulting model successfully captured the inherent imprecision and ambiguity in seismic datasets, particularly for varying magnitude scale. Therefore, in this section, the mathematical formulation will be defined to develop the T2FIBC model and then used to model the shoreline data. For a better understanding, each definition will be presented visually.

Definition 3.1

A T2FN is broadly defined as a type-2 fuzzy set that has a numerical domain. An interval of type-2 fuzzy set is defined using the following four constraints, where $\vec{A}_{\alpha} = \left\{\left[a^{\alpha}, b^{\alpha}\right], \left[c^{\alpha}, d^{\alpha}\right]\right\}$, $\forall \alpha \in [0,1]$, $\forall a^{\alpha}, b^{\alpha}, c^{\alpha}, d^{\alpha} \in {}^{\sim}$ (Figure 1):

1. $a^{\alpha} \le b^{\alpha} \le c^{\alpha} \le d^{\alpha}$
2. $[a^{\alpha}, d^{\alpha}]$ and $[b^{\alpha}, c^{\alpha}]$ generate a function that is convex and $[a^{\alpha}, d^{\alpha}]$ generate a normal function.
3. $\forall \alpha_1, \alpha_2 \in [0,1] : (\alpha_2 > \alpha_1) \Rightarrow \left(\left[a^{\alpha_1}, c^{\alpha_1}\right] \supset \left[a^{\alpha_2}, c^{\alpha_2}\right], \left[b^{\alpha_1}, d^{\alpha_1}\right] \supset \left[b^{\alpha_2}, d^{\alpha_2}\right]\right)$, for $c^{\alpha_2} \ge b^{\alpha_2}$.

4. If the maximum of the membership function generated by $[b^{\alpha},c^{\alpha}]$ is the level α_m, that is $[b^{\alpha_m},c^{\alpha_m}]$, then $\left[b^{\alpha_m},c^{\alpha_m}\right]\subset\left[a^{\alpha=1},d^{\alpha=1}\right]$[5, 23] .

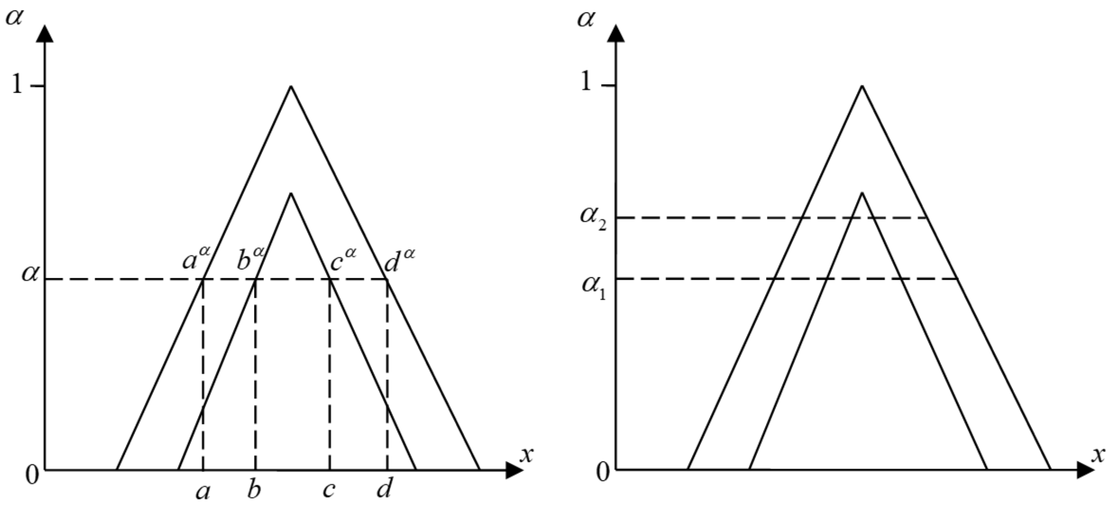

Fig. 1. Definition of an interval T2FN

Figure 1 shows that the illustration as the graphical representation of an interval T2FN. It visually explains the definition given in Definition 3.1, particularly how the fuzzy membership functions are bounded by intervals for each level α (alpha), and how constraints are respected in the structure of a T2FN .

Definition 3.2

Let $E=\{x|x \text{ type-2 fuzzy point}\}$ and $\ddot{\overleftrightarrow{E}}=\{E_i|E_i \text{ data point of earthquake magnitude}\}$ are type-2 fuzzy earthquake magnitude data (T2FEMD) with $E_i\in E\subset X$, where X is a universal set and $\mu_E(E_i):E\rightarrow[0,1]$ is the membership function defined as $\mu_E(E_i)=1$ and formulated as $\ddot{\overleftrightarrow{E}}=\left\{\left(E_i,\mu_E(E_i)\right)\middle|E_i\in{}^{\sim}, i=0,1,2,...,n\right\}$. Therefore,

$$\mu_E(E_i)=\begin{cases}0 & \text{if } E_i\notin X\\ c\in(0,1) & \text{if } E_i\,\ddot{\in}\,X\\ 1 & \text{if } E_i\in X\end{cases} \tag{1}$$

with $\mu_E(E_i)=\left\langle\mu_E\left(E_i^{\leftarrow}\right),\mu_E(E_i),\mu_E\left(E_i^{\rightarrow}\right)\right\rangle$ which $\mu_E\left(E_i^{\leftarrow}\right)$ and $\mu_E\left(E_i^{\rightarrow}\right)$ are left and right footprint of membership values with $\mu_E\left(E_i^{\leftarrow}\right)=\left\langle\mu_E\left(aE_i^{\leftarrow}\right),\mu_E\left(bE_i^{\leftarrow}\right)\right\rangle$ where, $\mu_E\left(aE_i^{\leftarrow}\right)$ and $\mu_E\left(bE_i^{\leftarrow}\right)$ are left-left, right-left membership grade values and $\mu_E\left(E_i^{\rightarrow}\right)=\left\langle\mu_E\left(cE_i^{\rightarrow}\right),\mu_E\left(dE_i^{\rightarrow}\right)\right\rangle$ where, $\mu_E\left(cE_i^{\rightarrow}\right)$and $\mu_E\left(dE_i^{\rightarrow}\right)$ are right-right, left-right membership grade values which can be written as

$$\ddot{\overleftrightarrow{E}}=\left\{\ddot{\overleftrightarrow{E}}_i : i=0,1,2,...,n\right\} \tag{2}$$

for every *i*, $\ddot{\overleftrightarrow{E}}_i=\langle[a_i,b_i]\rangle$ with $a_i=\langle[aa_i,bb_i]\rangle$ where aa_i and bb_i are left-left and right-left T2FEMD and $b_i=\langle[cc_i,dd_i]\rangle$ where cc_i and dd_i are left-right and right-right T2FEMD respectively. This can be illustrated as in Figure 2.

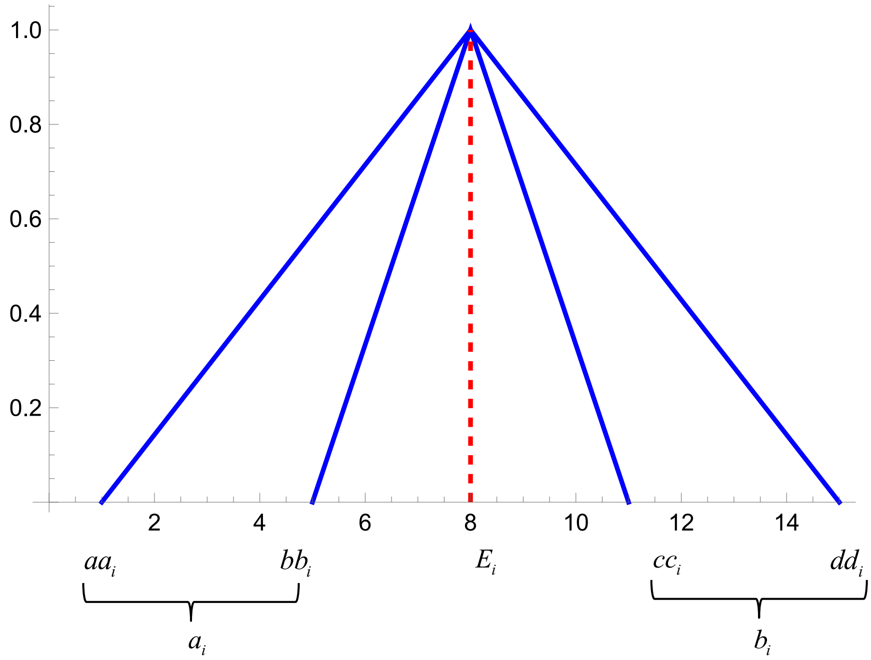

Fig. 2. T2FEMD around 8

Figure 2 provides a graphical illustration of a T2FEMD centered around a data point which is approximately at magnitude 8. This figure visually represents how uncertainty in both magnitude and membership is handled using type-2 fuzzy sets, where membership itself is fuzzy.

There are several different scales that seismologists use to measure the size of an earthquake. Local Magnitude (ML) as mentioned by Scordilis [29], Body-Wave Magnitude (mb) as discussed in [30], Surface-Wave Magnitude (Ms) [31], and Moment Magnitude (Mw) are the four types. Each of these scales is based on a different set of ideas, and they all have their own strengths and weaknesses. Because of this natural volatility, it is very hard to accurately record magnitudes. For example, the same earthquake might have several magnitude readings, with differences of ±0.2 to ±1.0 units.

The Local Magnitude scale often demonstrates reduced effectiveness when an earthquake registers a magnitude of M6.0 or greater. This indicates that significant events may not receive the recognition they warrant. The Surface-Wave Magnitude may not accurately reflect the characteristics associated with deep-focus earthquakes. The Body-Wave Magnitude is influenced by several factors, such as crust characteristics and frequency filtering, which can complicate the precision of magnitude reports. To effectively assess seismic data and understand the potential impacts of earthquakes, it is essential to recognize these differences.

A deficiency in confidence can pose challenges in the development of historical catalogs or in the examination of events that occurred within various tectonic contexts and networks. Researchers frequently employ conversion equations or probabilistic models to address this challenge. However, these methods may not adequately address both aspects of uncertainty, including potential variability in measurements and the ambiguity in scales.

At this time, T2FNs are pretty helpful. The T2FN can be applied the T2FN to represent both the range of potential magnitude values (like [6.2, 7.0]) and the fact that one is unsure about that range since the sources are unclear or disagree with each other. This is not the same as normal crisp or type-1 fuzzy models. For example, if both ML = 6.5 ± 0.5 and Ms = 6.8 ± 0.3 are reported for the same event, a T2FN may show this fuzzy set as a fuzzy envelope, which shows the greatest and lowest levels of confidence. This makes it less likely that the seismic model will make mistakes, which makes it easier to utilize Bézier curves to fill in the gaps and gain a better idea of seismic risk.

To analyze and compute with T2FN especially T2FEMD effectively, the concept of alpha-cut is often employed. An alpha-cut extracts an interval from the fuzzy set corresponding to a specific confidence level of alpha which helps to simplify the complex fuzzy envelope into manageable slices. In the context of earthquake magnitude modeling, using alpha-cuts allows researchers to systematically examine different levels of uncertainty within the reported seismic values. By analyzing these alpha-level intervals, one can better understand how uncertainty propagates and impacts risk assessments, leading to more robust and interpretable modeling, especially when integrating with methods like Bézier curve interpolation. Therefore, the following is the definition of the alpha-cut of T2FEMD.

Definition 3.3

Let $\ddot{\overleftrightarrow{E}}_i$ be the set of T2FEMDs with $\ddot{\overleftrightarrow{E}}_i \in \ddot{\overleftrightarrow{E}}$ where $i = 0,1,\ldots,n-1$. Then $\ddot{\overleftrightarrow{E}}_{i_\alpha}$ is the alpha-cut operation of T2FEMDs with $i = 0,1,2,\ldots,n$ which is given as follows.

$$
\begin{aligned}
\ddot{\overleftrightarrow{E}}_{i_\alpha} &= \left\langle a_{i_\alpha}, E_i, b_{i_\alpha} \right\rangle \\
&= \left\langle \left[aa_{i_\alpha}, bb_{i_\alpha} \right], E_i, \left[cc_{i_\alpha}, dd_{i_\alpha} \right] \right\rangle \\
&= \left\langle \left(E_i - \left[aa_{i_\alpha}, bb_{i_\alpha} \right] \right)\alpha + \left[aa_{i_\alpha}, bb_{i_\alpha} \right], E_i, -\left(\left[cc_{i_\alpha}, dd_{i_\alpha} \right] - E_i \right)\alpha + \left[cc_{i_\alpha}, dd_{i_\alpha} \right] \right\rangle
\end{aligned}
\tag{3}
$$

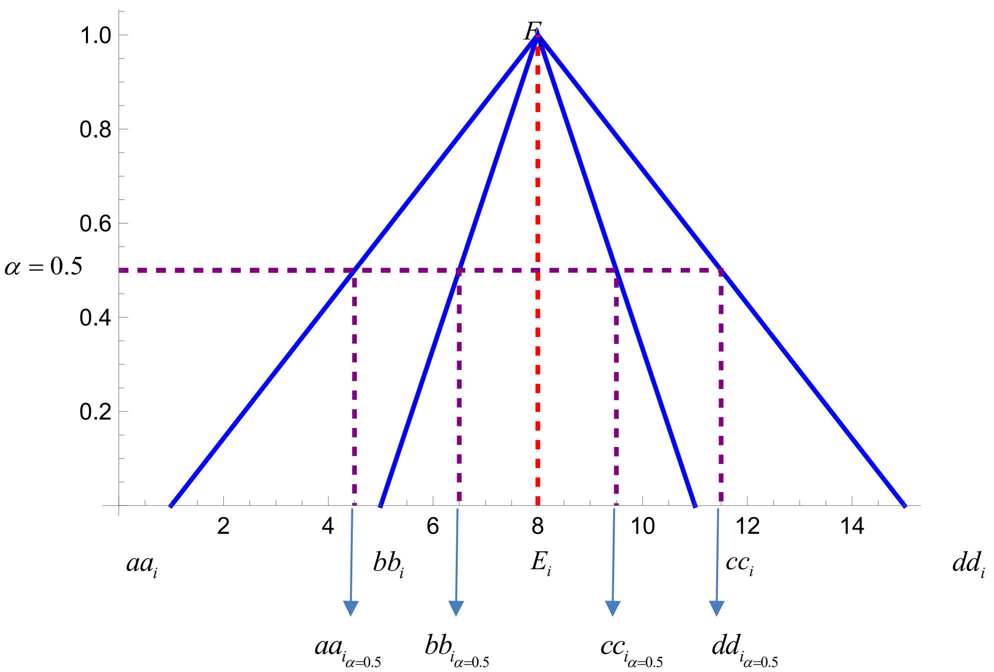

Fig. 3. The alpha-cut operation towards T2FEMD.

Fig. 3. Visually illustrates the alpha-cut operation applied to a T2FEMDrepresented by a triangular T2FN. The figure shows how the alpha-level (in this case, α = 0.5) is used to extract the core values from a type-2 fuzzy membership function.

In the T2FEMD architecture, picking the alpha value (α) is a critical step to make sure that uncertainty is always understood in a usable manner. To estimate the alpha level, this study looks at the centroid of the triangular T2FEMD. The centroid is the center of mass or point of equilibrium of the triangular fuzzy area. It displays the amount of confidence that best represents all of the fuzzy members. The alpha-cut operation can properly display the average level of uncertainty in the data when α is set to the centroid height. This provides a logical basis for determining the interval bounds that will be used in the future steps of type reduction and defuzzification. Therefore, the equation for obtaining the value of alpha based on triangular T2FEMD can be given as follows based on Figure 2.

$$\alpha^c = \frac{1}{6}\left(aa_i + F_i + dd_i + bb_i + F_i + cci\right) \tag{4}$$

The type-reduction method is the next important step in using T2FNs after the alpha-cut. To make a type-1 fuzzy set out of a T2FN, type-reduction combines the interval information from different alpha-levels. This approach helps to combine the ambiguity in the upper and lower membership functions into one understandable form that is ready to be defuzzified. Type-reduction makes it possible to get a representative magnitude range such as an average or centroid, from seismic data modeling. This is because the T2FEMD captures the dispersion and imprecision of the data. This phase is very important before you make any final guesses or choices based on hazy earthquake data. The type-reduction procedure for T2FEMD can be defined formally below.

Definition 3.4

Let $\ddot{\vec{E}}_i$ be a set of T2FEMD and $\ddot{\vec{E}}_{i_{\alpha^c}}$ are the set of T2FEMD after the alpha-cut process for $i = 0,1,2,..,n$, then the type-reduction of $\ddot{\vec{E}}_{i_{\alpha^c}}$ which is represented as $\ddot{\vec{E}}^{TR}_{i_{\alpha^c}}$ can be defined as follows

$$\ddot{\vec{E}}^{TR}_{\alpha^c} = \left\{ \ddot{\vec{E}}^{TR}_{i_{\alpha^c}} = \left\langle a^{TR}_{i_{\alpha^c}}, E_i, b^{TR}_{i_{\alpha^c}} \right\rangle \middle| i = 0,1,2,...,n \right\} \tag{5}$$

where E_i is crisp data points and $a^{TR}_{i_{\alpha^c}}$ and $b^{TR}_{i_{\alpha^c}}$ are left and right type-reduced alpha-cut T2FEMD respectively with their formulation given by

$$\begin{aligned} a^{TR}_{i_{\alpha^c}} &= \frac{1}{2}\sum_{i=0,1,...,n} \left\langle aa_{i_{\alpha^c}} + bb_{i_{\alpha^c}} \right\rangle \\ b^{TR}_{i_{\alpha^c}} &= \frac{1}{2}\sum_{i=0,1,...,n} \left\langle cc_{i_{\alpha^c}} + dd_{i_{\alpha^c}} \right\rangle \end{aligned} \tag{6}$$

The interval $\left[a^{TR}_{\alpha^c}, b^{TR}_{\alpha^c} \right]$ indicates the range of uncertainty for each alpha-level in the T2FEMD set once the type-reduction method is finished. The penultimate step, termed defuzzification, is done to acquire a single clear value that represents the whole range. Defuzzification changes the interval type-reduced result into a single number that you may use straight away to make choices or perform further modeling. This is particularly important when figuring out how likely an earthquake is to happen, as you usually need a very good idea of how big it will be even if there are some unknowns. The official definition of defuzzification that is used in this context is below.

Definition 3.5

Let $\overleftrightarrow{\overleftrightarrow{E}}{}^{TR}_{i_{\alpha^c}}$ be the type-reduction alpha-cut T2FEMD with $i = 0,1,2,...,n$.Then, $\overleftrightarrow{\overleftrightarrow{E}}{}^{D}_{i_{\alpha^c}}$ is defuzzification process of $\overleftrightarrow{\overleftrightarrow{E}}{}^{TR}_{i_{\alpha^c}}$ if for every $\overleftrightarrow{\overleftrightarrow{E}}{}^{TR}_{i_{\alpha^c}} \in \overleftrightarrow{\overleftrightarrow{E}}{}^{TR}_{\alpha^c}$,

$$\overleftrightarrow{\overleftrightarrow{E}}{}^{D}_{\alpha^c} = \left\{ \overleftrightarrow{\overleftrightarrow{E}}{}^{D}_{i_{\alpha^c}} \middle| i = 0,1,2,...,n \right\} \tag{7}$$

where for each $\overleftrightarrow{\overleftrightarrow{E}}{}^{D}_{i_{\alpha^c}}$ can be formalized as

$$\overleftrightarrow{\overleftrightarrow{E}}{}^{D}_{i_{\alpha^c}} = \frac{1}{3} \sum_{i=0,1,...,n} \left\langle a^{TR}_{i_{\alpha^c}}, E_i, b^{TR}_{i_{\alpha^c}} \right\rangle \tag{8}$$

The second-to-last step in turning the T2FEMD from a type-reduction alpha-cut representation $\left[a^{TR}_{\alpha^c}, b^{TR}_{\alpha^c} \right]$ into a clear value Definition 3.5 says that $\overleftrightarrow{\overleftrightarrow{E}}{}^{D}_{i_{\alpha^c}}$ is the approach to get rid of fuzziness. To do this, it needs to determine the average of the lower limit, centroid, and upper bound of the type-reduced fuzzy triple. This makes sure that the uncertainty is expressed in a fair manner. The defuzzified values are very important since they provide you with exact points to utilize to go on to the next step in geometric modeling. The next step is to create a cubic Type-2 fuzzy interpolation Bézier curve, and these numbers go straight into it. At this point, the statistics are used to create smooth, continuous curves that show how seismic data behaves when it's not obvious. Defuzzification fills in the spaces between showing fuzzy uncertainty and filling in geometric shapes. This makes sure that the final image is smooth, correct, and easy to understand.

The next important step is to use the defuzzified points to construct a geometric model that is both structured and adaptable. This makes the process of defuzzification longer. This creates a piecewise cubic type-2 fuzzy interpolation Bézier curve that enables you to change the curve segments in a specific area while keeping the transitions between them smooth. By linking together many Bézier segments that change with the data, this method makes it possible to display seismic data or other datasets that are uncertain in a way that is useful. You may see the official definition of this technique in Definition 3.6.

Definition 3.6

Given T2FEMD, $\overleftrightarrow{\overleftrightarrow{E}}_i$ and type-2 fuzzy derivative values at t, $\overleftrightarrow{\overleftrightarrow{D}}_i$ where $\overleftrightarrow{\overleftrightarrow{D}}_i, \overleftrightarrow{\overleftrightarrow{E}}_i \in \sim$, $i = 0,1,...,n$. Then, the T2FIBC can be defined as

$$\overleftrightarrow{\overleftrightarrow{B}}(t) = (1-t)^3 \overleftrightarrow{\overleftrightarrow{E}}_i + 3t(1-t)^2 \overleftrightarrow{\overleftrightarrow{K}}_i + 3t^2(1-t) \overleftrightarrow{\overleftrightarrow{L}}_i + t^3 \overleftrightarrow{\overleftrightarrow{E}}_{i+1} \tag{9}$$

with

$$\begin{aligned} \overleftrightarrow{\overleftrightarrow{K}}_i &= \frac{\overleftrightarrow{\overleftrightarrow{D}}_i}{3} + \overleftrightarrow{\overleftrightarrow{E}}_i \\ \overleftrightarrow{\overleftrightarrow{L}}_i &= \overleftrightarrow{\overleftrightarrow{E}}_{i+1} - \frac{\overleftrightarrow{\overleftrightarrow{D}}_{i+1}}{3} \end{aligned} \tag{10}$$

such that $\overleftrightarrow{\overleftrightarrow{D}}_i$ and $\overleftrightarrow{\overleftrightarrow{D}}_{i+1}$ are type-2 tangent vector of T2FEMD where the representation of the type-2 fuzzy tangent values can be given as follows.

$$\overleftrightarrow{\overleftrightarrow{D}}_0 = 2\left(\overleftrightarrow{\overleftrightarrow{E}}_1 - \overleftrightarrow{\overleftrightarrow{E}}_0 \right) - \frac{\left(\overleftrightarrow{\overleftrightarrow{E}}_2 - \overleftrightarrow{\overleftrightarrow{E}}_0 \right)}{2}, \tag{11}$$

$$\overset{\leftrightarrow}{\ddot{D}}_n = 2\left(\overset{\leftrightarrow}{\ddot{E}}_n - \overset{\leftrightarrow}{\ddot{E}}_{n-1}\right) - \frac{\left(\overset{\leftrightarrow}{\ddot{E}}_n - \overset{\leftrightarrow}{\ddot{E}}_{n-2}\right)}{2}, \tag{12}$$

$$\overset{\leftrightarrow}{\ddot{D}}_i = h_i\left(\overset{\leftrightarrow}{\ddot{E}}_i - \overset{\leftrightarrow}{\ddot{E}}_{n-1}\right) + \left(1-h_i\right)\left(\overset{\leftrightarrow}{\ddot{E}}_{i+1} - \overset{\leftrightarrow}{\ddot{E}}_i\right) \tag{13}$$

where

$$h_i = \frac{\left\|\overset{\leftrightarrow}{\ddot{E}}_{i+1} - \overset{\leftrightarrow}{\ddot{E}}_i\right\|}{\left\|\overset{\leftrightarrow}{\ddot{E}}_{i+1} - \overset{\leftrightarrow}{\ddot{E}}_i\right\| + \left\|\overset{\leftrightarrow}{\ddot{E}}_i - \overset{\leftrightarrow}{\ddot{E}}_{i-1}\right\|} \tag{14}$$

for $i = 1, 2, .., n-1.$

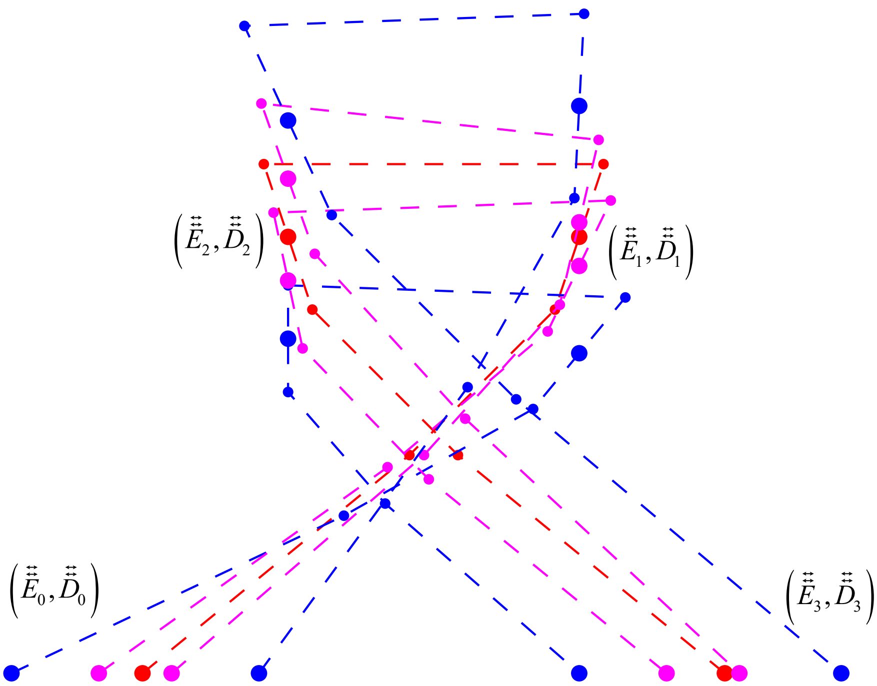

Fig. 4. Type-2 fuzzy control polygon consists of type-2 fuzzy tangent vector and T2FEMD

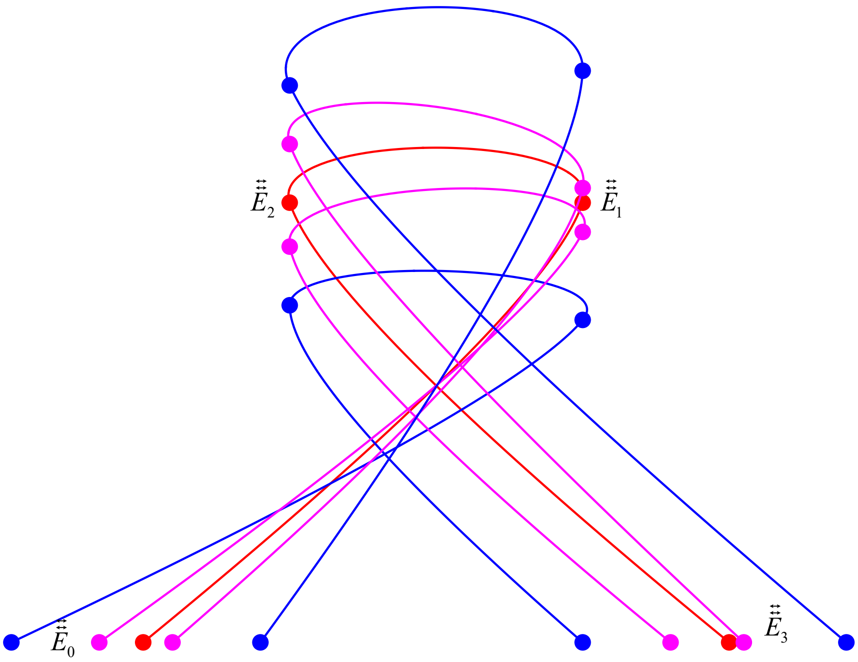

Fig. 5. T2FIBC interpolates all T2FEMD, $\overset{\leftrightarrow}{\ddot{E}}_0, \overset{\leftrightarrow}{\ddot{E}}_1, \overset{\leftrightarrow}{\ddot{E}}_2$ and $\overset{\leftrightarrow}{\ddot{E}}_3$.

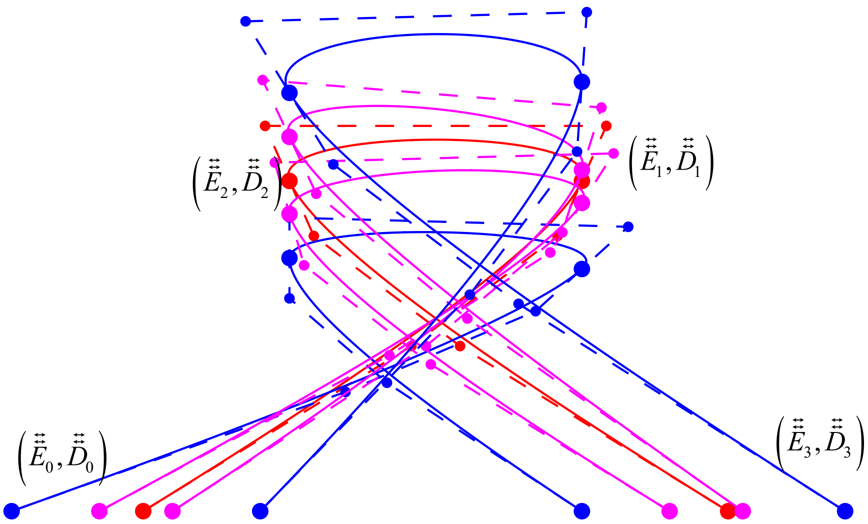

Fig. 6. T2FIBC interpolates all T2FEMD together with the type-2 fuzzy control polygon and the type-2 fuzzy tangent vector.

Figures 4, 5, and 6 show how to use the T2FEMD and the type-2 fuzzy tangent vectors that go along with it to build a T2FIBC model. Figure 4 shows the type-2 fuzzy control polygon. The type-2 fuzzy derivative $\overset{\leftrightarrow}{\ddot{D}}_i$ of each T2FEMD , $\overset{\leftrightarrow}{\ddot{E}}_i$ tells the interpolation which way to go. The dashed lines and highlighted regions between the control points and tangent vectors illustrate the ambiguity in the data and its direction. Now in Figure 5, thus the interpolation is the main thing. The T2FIBC is a smooth cubic Bézier curve that connects all the T2FEMD $\overset{\leftrightarrow}{\ddot{E}}_i = \langle a_i = [aa_i, bb_i], E_i, b_i = [cc_i, dd_i] \rangle$, showing how unsure each point is. The light and dark areas of the type-2 fuzzy curve show where it starts and ends. These pieces show how much the fuzzy set may change. Lastly, Figure 6 displays the whole type-2 fuzzy control polygon, the T2FIBC, and the type-2 fuzzy tangent vectors all in one picture. This close-up picture shows how the curve follows the type-2 fuzzy data while being directed by the type-2 fuzzy tangents. This is a better way to exhibit data patterns that aren't very apparent or are hard to see.

3.1 Type-2 Fuzzy Interpolation Bezier Curve of Earthquake Magnitude Data at Lahad Datu, Sabah

There have been numerous earthquakes in Lahad Datu, Sabah, with 28 occurring in 2017 alone. The area's seismic activity is attributed to its proximity to active fault lines, which cause earthquakes in both the area and the region. Along with Ranau and Kudat, Lahad Datu is one of Malaysia's most seismically active areas. This makes it more susceptible to being affected by earthquakes, especially those with high magnitudes as discussed in a previous study [32,33].

In 2017, a study found that Lahad Datu experienced numerous small tremors. This highlights the importance of being aware of the seismic risks in the area. On July 26, 1976, a significant earthquake occurred in the area with a moment magnitude of 5.8. This suggests that it might experience even greater earthquakes in the future as mentioned in [32,33]. On June 5, 2015, a 6.0 magnitude earthquake happened in Ranau. This was a big deal and showed how active the geology is in the region. It has to be looked at [35].

To determine the safety of structures, researchers have developed models that estimate the peak ground acceleration (PGA) at a specific location. According to a study, more than 400 earthquakes with magnitudes between 2.2 and 6.0 happened in Lahad Datu between 1900 and 2017. This means that Lahad Datu has a high seismic hazard index which is stated in [34, 36]. Because of this sort of knowledge, planning for disasters and ways to lessen their effects needs to be very careful and take the state's geology into account.
These earthquakes impact how cities are designed and how strong their infrastructure is. Studies reveal that even small earthquakes may damage many buildings in Sabah, notably in Lahad Datu. We need to be ready for disasters and work to make things better. Local and regional governments need to do more to lower the risk of catastrophes so that they are ready for future earthquakes.

In short, the earthquake magnitude data for Lahad Datu shows that there is a constant seismic hazard since there are regular low to moderate seismic events. This means that disaster risk management techniques need to be in place all the time.

To better understand and model the uncertainty within the earthquake magnitude data of Lahad Datu, especially in light of the area's continuous seismic activity, advanced mathematical tools are necessary. One such tool is the T2FIBC, which offers a robust approach to capture and represent imprecise or conflicting data points across time. This method allows for a smooth and flexible interpolation of seismic magnitude values while preserving the uncertainty embedded in the original data. In this context, Definition 3.6 is applied to construct the T2FIBC, using the T2FEMD and the associated type-2 fuzzy derivatives to define the curve’s behavior and tangent directions accurately.

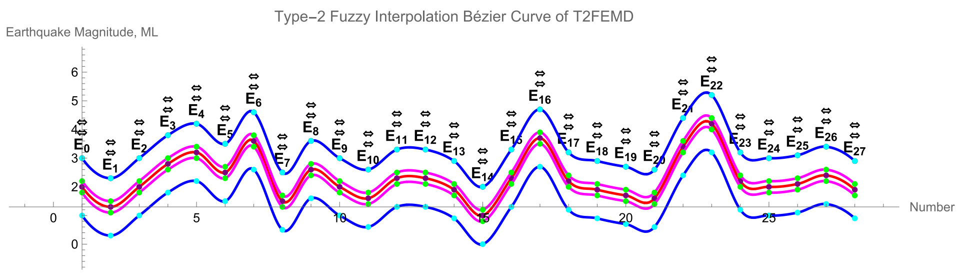

Fig. 7. Piecewise cubic T2FIBC of T2FEMD

Figure 7 illustrates the piecewise cubic type-2 fuzzy interpolation Bézier curve. The data about the size of earthquakes in Lahad Datu was utilized. The data presents challenges in comprehension due to the frequent earthquakes that occurred in the area throughout 2017. The external blue curves in the model illustrate the application of T2FEMD in generating upper and lower footprints of uncertainty. These curves describe the potential variations in the magnitude of an earthquake. Colored interpolation Bézier curves are illustrated, representing the lower footprints of uncertainty. The green points indicate the T2FEMD corresponding to the lower and upper bounds of uncertainty employed in constructing these curves. The purple points represent the actual data collected from the earthquake. According to Definition 3.6, this approach ensures that the uncertain data is interpolated smoothly and continuously, while also taking into account the inherent fuzziness of the original measurements. The graphic effectively depicts the complete uncertainty profile over time, featuring indistinct segments between the upper and lower limits.

The alpha-cut operation defined in Definition 3.3 will be utilized to derive horizontal slices of the T2FEMD, represented as T2FN, at specific confidence levels. This analysis aims to examine the uncertain earthquake magnitude data with greater precision. The alpha-values in this instance are not selected arbitrarily or uniformly. Rather than, they are determined by the centroid (center of mass) of the triangle T2FN, as articulated in Eq. (4), which serves as the point of balance for each data segment. This method of selecting alpha according to the centroid is effective and data-driven, ensuring an optimal balance in uncertainty representation. The alpha-cuts offer a collection of intervals utilized for constructing intermediate Bézier curves within the framework of the footprint of uncertainty. The curves illustrate the transition from complete uncertainty to heightened confidence. The centroid represents the average impact of the fuzzy region. Consequently, these curves demonstrate both mathematical precision and physical relevance. Figure 8 illustrates the curves derived from the centroid-based alpha-values. The interpolation demonstrates the effectiveness of the alpha-cut strategy in providing the most balanced representation of uncertainty.

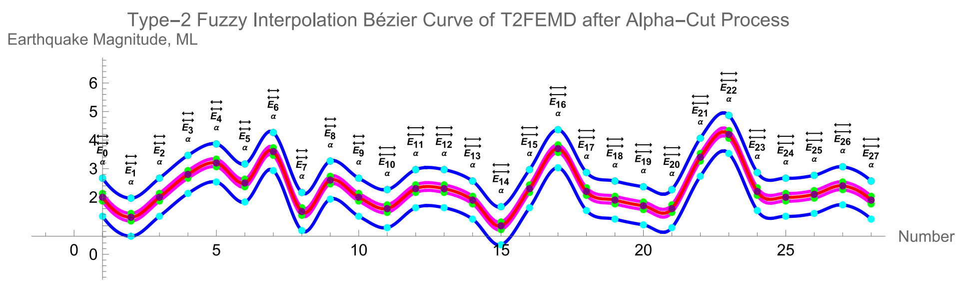

Fig. 8. Piecewise cubic T2FIBC of T2FEMD after alpha-cut process.

Figure 8 illustrates the outcomes derived from applying the alpha-cut operation (Definition 3.3) to all 28 data points in the T2FEMD, employing a constant alpha value of 0.3333. The specified number represents the centroid, or center of mass, of a standard triangular fuzzy number, indicating the position where the distribution is uniform across all intervals. All 28 T2FEMD intervals exhibit a uniform width and show a symmetrical pattern. Employing a fixed alpha value of 0.3333 ensures a consistent and precise representation throughout all type-2 fuzzy data segments. The curves illustrated in Figure 8 showcase a variety of intermediate Bézier interpolations that reside within the footprint of uncertainty. However, they do not extend to the entire boundary. The curves accurately illustrate uncertainty by presenting both the range and the central tendency of the fuzzy model. The data indicate a decrease in uncertainty alongside an increase in confidence at this alpha level. The effectiveness of the centroid-based α value in uniform data segments is demonstrated, highlighting its ability to uphold both mathematical consistency and physical interpretability.

Following the alpha-cut operation, the type-reduction procedure represents the subsequent critical step in the processing of T2FEMD as outlined in Definition 3.4. The objective of this method is to convert a T2FEMD into type-1 fuzzy earthquake magnitude data, which can then be transformed into a clear output. Type reduction is crucial in this context as it enhances the comprehensibility and applicability of the T2FEMD's layered uncertainty, particularly when making decisions or analyzing data. The type-reduction approach examines all alpha-level intervals, particularly those with values such as alpha equals 0.3333, and identifies a weighted representative interval or point that reflects the overall uncertainty. This version maintains the essential ambiguity while enhancing usability through the incorporation of curves or values that are type-reduced. This indicates that seismic modeling does not consistently provide accurate earthquake magnitudes, and it also highlights that the data is not always flawless.

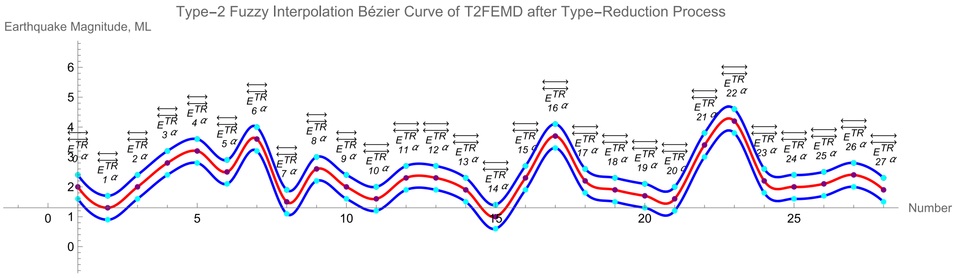

Fig. 9. Piecewise cubic T2FIBC after type-reduction process.

Figure 9 presents the result of applying the type-reduction process to the type-2 fuzzy interploation Bezier curve model for the T2FEMD. After conducting the alpha-cut process at meaningful confidence levels (such as α = 0.3333), the type-reduction operation aggregates the resulting fuzzy intervals into type-1 fuzzy values that better represent the average or overall uncertainty of the system. The figure illustrates how the previously layered fuzzy Bézier curves are now reduced to a single representative curve, shown in red. Meanwhile, the blue boundary curves still indicate the full extent of the original uncertainty (i.e., the footprint of uncertainty). This type-reduced curve lies centrally within the uncertainty footprint. It retains the essential trends of the fuzzy model but in a form that is more interpretable and usable for further analysis or decision-making. It highlights how uncertainty is not eliminated but rather summarized, allowing seismic modelers to assess expected magnitudes under uncertainty without ignoring variability.

Defuzzification is the final stage in processing the T2FEMD, as mentioned in Definition 3.5. This only occurs once when you reduce the number of sorts. Defuzzification converts the type-reduced fuzzy output into a clear value that can be used immediately in real-life applications, such as preparing for disasters, building infrastructure, or setting up early warning systems. Defuzzification is particularly significant in this scenario, as it provides a clear indication of the earthquake's was by considering all the fuzzy layers and uncertainties introduced in the previous steps. The model determines how to proceed, which may involve finding the centroid, the average of the maxima, or the mean of the interval endpoints. This step is crucial for the fuzzy model to function effectively in real-life applications. The final answer should be based on math, but it should also make sense in the actual world. Defuzzification turns the type-2 fuzzy earthquake magnitude from a complex fuzzy representation into a single value that stands for everything. This makes it easy to interpret, distribute, and use information on seismic risk.

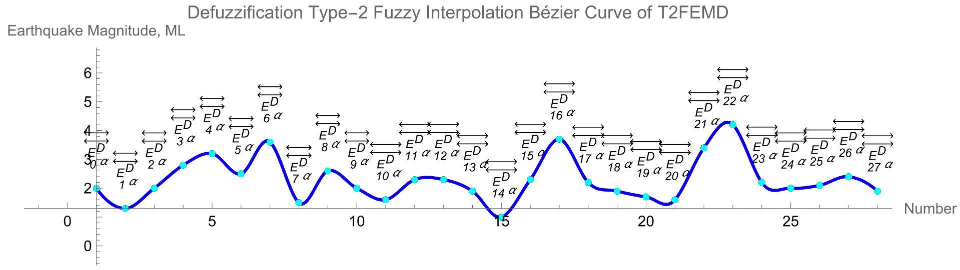

Fig. 10. Defuzzified T2FIBC after type-reduction process.

Figure 10 illustrates the defuzzified T2FIBC, representing the concluding phase in the type-reduction and defuzzification process for the earthquake magnitude data. The curve establishes a clear and straightforward path from a dataset that was previously unclear and difficult to understand. It effectively demonstrates the fundamental seismic trend. The defuzzified curve is positioned centrally between the upper and lower limits of the fuzzy envelope. It offers a metric that quantifies the magnitude of earthquakes while considering all the uncertainties inherent in the fuzzy model. The labels on the curve, $\overset{\leftrightarrow}{\ddot{E}}{}^{TR}_{i_\alpha}$ indicate the points at which the fuzzy model underwent initial simplification followed by defuzzification, $\overset{\leftrightarrow}{\ddot{E}}{}^{D}_{i_\alpha}$. This assigns a singular representative value to each data index. This output facilitates decision-making in areas such as disaster preparedness and infrastructure development, where accurate and reliable data is essential.

The comparison between the original earthquake magnitude data points of the interpolated Bezier curve and the defuzzified T2FIBC indicates that there is minimal difference between the two. The defuzzified curve aligns closely with the pattern and shape of the original data, preserving both the trend and the range of the seismic observations. This close match indicates that the model effectively captures the fundamental behavior of earthquake magnitudes, despite employing fuzzy logic to incorporate uncertainty. The slight variation indicates that the fuzzy interpolation process, along with type-reduction and defuzzification, preserves the essential information while enhancing it through a systematic approach to uncertainty. This enhances the model's effectiveness in earthquake analysis and disaster risk planning, where precise yet robust representations of uncertain data are crucial.

Figure 11 shows both the original data curve and the defuzzified T2FIBC. This makes it easy to see how closely the predicted values align with the actual values generated. The picture makes it extremely clear that the two curves almost touch at every point. This shows how accurate and reliable the fuzzy interpolation and defuzzification technique is in modeling real-world earthquake magnitude data.

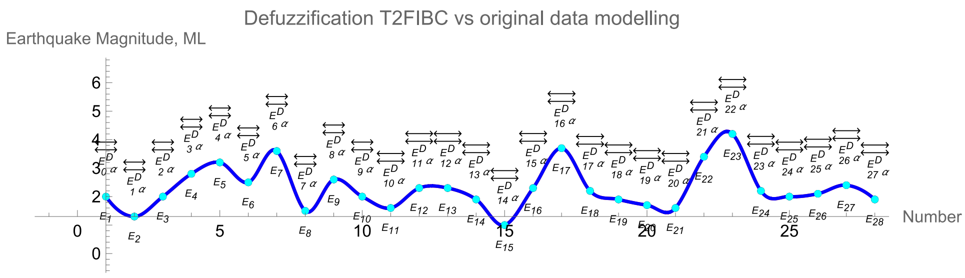

Fig. 11. Defuzzified T2FIBC model against crisp interpolation Bezier curve model.

Figure 11 presents the defuzzified T2FIBC alongside the original data curve. This serves as a prime illustration of the effectiveness of the fuzzy modeling approach. The blue defuzzified curve in the image appears to align closely with all of the original data points. At nearly every observation point, the two curves are almost indistinguishable from one another. The modeled data demonstrates a notable similarity to the real data. This visual agreement indicates that the fuzzy interpolation and defuzzification process maintains the integrity of the original data while effectively addressing and incorporating uncertainty. The defuzzified T2FIBC accurately reflects the genuine seismic trends, eliminating any extraneous noise. This demonstrates that the model is effective and performs its intended functions accurately. This statistic supports the assertion that the recommended technique is especially suitable for modeling real-world earthquake magnitude data, where both accuracy and the ability to manage uncertainty are essential.

Basic statistical methods, including RMSE (Root Mean Square Error), MAE (Mean Absolute Error), MAPE (Mean Absolute Percentage Error), and R^2 (Coefficient of Determination), will be employed to evaluate the model's performance and accuracy. The provided figures indicate the proximity of the defuzzified Bézier model to the actual magnitudes of significant earthquakes. The accuracy of the model's prediction can be assessed using RMSE. The situation deteriorates further when it becomes more severe. MAE, conversely, indicates the average deviation of each piece of information from the actual values. The mistake rate for MAPE remains consistent, regardless of the size. This facilitates comprehension. The R^2 score demonstrates the model's ability to elucidate the differences in the actual data. A value near 1 indicates that the model aligns closely with the data. Numerous pieces of evidence indicate that the model is robust in demonstrating complex earthquake patterns while remaining user-friendly and comprehensible. The model has been validated and can be utilized to enhance earthquake monitoring and improve safety measures. The assessment included a variety of distinct indicators. The outcomes of all four statistical methods are summarized in Table 1.

Table 1
Statistical methods analyses between model data and actual data

Statistical Method	Formula	Result (Model Data vs Actual Data)
Root Mean Square Error (RMSE)	$RMSE = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(y-\hat{y}_i\right)^2}$	1.6785×10^{-16}
Mean Absolute Error (MAE)	$MAE = \frac{1}{n}\sum_{i=1}^{n}\lvert y_i-\hat{y}_i\rvert$	7.93016×10^{-17}
Mean Absolute Percentage Error (MAPE)	$MAPE = \frac{100}{n}\sum_{i=1}^{n}\left\lvert\frac{y_i-\hat{y}_i}{y_i}\right\rvert$	3.41238×10^{-15} %
Coefficient of Determination (R^2)	$R^2 = 1-\frac{\sum\left(y_i-\hat{y}_i\right)^2}{\sum\left(y_i-\bar{y}\right)^2}$	1

Four statistical tests were conducted on the defuzzified T2FIBC model utilizing RMSE, MAE, MAPE, and R^2 to assess its accuracy and performance effectiveness. The results are comprehensively outlined in Table 1. The statistical methods demonstrate the degree of alignment between the model's output and the actual data regarding earthquake magnitudes. The findings indicate that the model reveals a notable level of accuracy, as evidenced by the low error values observed. The RMSE value indicates a significant relationship between the model's predictions and the actual data. This shows that the variations hold minimal significance. The MAE value indicates a slight average absolute discrepancy between the model and the actual data. This offers additional validation of the model's precision.

The MAPE score indicates that the model maintains a high level of accuracy, even when handling extensive datasets. The model demonstrates a perfect fit, with the R^2 value precisely at 1. This indicates that it successfully addresses all discrepancies identified in the actual data. The model successfully aligns with the primary trend while also recognizing nuanced differences in earthquake magnitudes. The compiled data indicate that the defuzzified model serves as an effective tool for understanding the mechanics of earthquakes in real-world scenarios. This strategy showcases considerable effectiveness by providing a systematic method for analyzing intricate data, which can be employed for tracking earthquakes and informing decision-making processes.

4. Discussion

The T2FIBC model offers a reliable approach for conveying data that may be ambiguous, such as the intensity of earthquakes. Type-1 fuzzy and regular crisp models frequently fall short in effectively capturing the degree of uncertainty and ambiguity present in real-world scenarios, including earthquakes. Conventional methods often demonstrate reduced efficacy when applied to complex real-world datasets, as they fail to address secondary uncertainty adequately. This highlights the inherent ambiguity associated with the membership function.

The incorporation of T2FNs, along with a footprint of uncertainty, effectively addresses these challenges within the T2FIBC model. This footprint of uncertainty demonstrates the range of values you may possess (primary membership) and the intensity of your belief in specific values (secondary membership). This introduces additional complexity to the modeling environment and broadens the array of available options. The implementation of triangular T2FNs as control points enhances the clarity of curve design and accelerates the computational process.

The modeling method consists of three primary processes: fuzzifying the control points, employing alpha-cuts to construct the T2FIBC and concluding with type-reduction and defuzzification. The alpha-cut method helps identify the curve with varying degrees of certainty. This is essential for modeling geophysical uncertainty, as decision-makers typically seek to understand a range of possible outcomes. This piecewise interpolation curve is subject to modification within a specified area. The rapid changes in the dataset or the emergence of atypical patterns enhance the model's flexibility.

The precise information concerning the intensity of the earthquake and the inaccurate T2FIBC model were identical. They exhibited a notable degree of similarity. This demonstrates that the model can effectively replicate the observed data while preserving all essential components. The statistical performance indicators that further supported this included the Root Mean Square Error (RMSE), the Mean Absolute Error (MAE), the Mean Absolute Percentage Error (MAPE), and the coefficient of determination (R^2). The model demonstrates remarkable precision in forecasting object dimensions, as indicated by the minimal RMSE and MAE values. A model exhibiting a high R^2 score clearly accounts for a significant portion of the variations observed in the data.

The examination of seismological data is enhanced as the model accurately defines the upper and lower limits of the footprint of uncertainty, thus emphasizing the related degree of uncertainty. The specified parameters serve as structures that illustrate the broader context of the predictions. This process is crucial for identifying potential threats, improving urban development, and reducing the likelihood of adverse events. The findings indicate that the T2FIBC serves not only as a curve-fitting technique but also as a comprehensive modeling framework that is mathematically sound, easily comprehensible, and effectively demonstrates uncertainty.

The model demonstrates versatility for application across various scenarios. This research focused primarily on seismic magnitude data. This methodology can also be applied across various fields where data may be ambiguous or significantly imprecise, including climate predictions, financial time series, or biological metrics.

5. Conclusion

This study has employed a T2FIBC model to demonstrate the inaccuracies in seismic data regarding the magnitude of earthquakes. Incorporating T2FNs into the interpolation framework enhances the handling of data that may not be entirely accurate. This will facilitate a comprehensive understanding of seismic data's functioning.

The model effectively demonstrates patterns and variations within the dataset and is capable of handling various degrees of uncertainty. This represents a significant advancement over current systems, as it effectively addresses uncertainty in both data values and member decisions.

The T2FIBC model serves as an effective method for elucidating the significance of geoscientific data for those who may be uncertain about its meaning. It can serve purposes beyond merely measuring earthquakes and can also be applied when the data is complex and multidimensional. In the future, it could be advantageous to utilize hybrid fuzzy-possibilistic frameworks or integrate this model with machine learning techniques.

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