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Hesitant Fuzzy B-Spline Modeling: A New Approach to Quintic Spatial Curve Approximation Model

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ABSTRACT

Fuzzy geometric modeling has been applied in various fields, where the fuzzy set approach allows the representation of uncertainty for better visualization. In fact, numerous types of fuzzy sets, including the traditional fuzzy set, type-2 fuzzy set, and intuitionistic fuzzy set, among others, are successfully applied in geometric modeling, while there is a gap in representing hesitancy, which often occurs during data collection. The hesitant fuzzy set provides a method to represent hesitancy—in other words, all opinions on the data through different membership degrees assigned to each element. This paper aims to introduce a novel model, namely the hesitant fuzzy B-spline curve approximation model. The study starts with a comprehensive literature review on the development of fuzzy geometric modeling. By applying the fundamental concepts of the hesitant fuzzy set and hesitant fuzzy control point relation defined, the hesitant fuzzy B-spline curve approximation model is constructed. A numerical example of hesitant fuzzy quintic B-spline spatial curves is visualized through the model. The study demonstrates an effective and flexible method to represent uncertainty through hesitancy, which will advance the field of fuzzy geometric modeling.

1. Introduction

Geometric modeling has played a key role in driving modern technological progress across many fields. Since its early days, it has been essential for representing complex shapes and surfaces in a mathematical way through parametrization, a concept first introduced by Pierre Bézier in the 1960s [1]. The history of curve modeling dates back to the foundation work of Pierre Bézier and Paul de Casteljau in the 1940s and 1950s. They developed Bézier curves, which were originally used in car design [2]. Around the same time, the groundwork for spline functions was introduced, which later became one of the important techniques for curve approximation [3]. As computational geometry advanced in the 1970s, researchers introduced B-spline curves [4] and followed by B-spline surface afterwards. In the 1980s, Dale Myers and John Hart introduced NURBS, which allowed for greater flexibility in curve modeling [5]. Today, geometric modeling is widely used in areas like engineering,

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computer graphics, and data visualization [6,7]. Its versatility makes it possible to design and analyze complicated structures, from car parts to buildings, fueling innovation in many industries.

On the other hand, the concept of fuzzy set theory was introduced by Lotfi A. Zadeh in 1965, which aims to address the uncertainty in decision-making. Fuzzy sets deal with imprecise information by enabling its elements to have membership degrees instead of just yes or no classification [8]. Afterwards, plenty of fuzzy sets are introduced, including type-2 fuzzy sets to handle the uncertainty of membership degree itself [9], intuitionistic fuzzy sets to represent non-membership degree and uncertainty degree [10], interval-valued fuzzy sets, picture fuzzy sets, neutrosophic fuzzy sets and so on. Every fuzzy set stated above deals with single membership degree for each element. In 2000s, Torra and Narukawa introduced a novel fuzzy set, namely hesitant fuzzy set. This new approach allowed for multiple membership values to reflect the hesitation or uncertainty people often have when making decisions [14]. Hesitant fuzzy sets made it easier to represent situations where decision-makers are unsure, offering more flexibility than traditional fuzzy sets.

Over the years, researchers have explored combining fuzzy set theory with geometric modeling. In 1995, Anile and his team used fuzzy arithmetic to manage uncertain data in environmental impact studies [15]. Afterwards, researchers developed plenty of geometric modeling models by using fuzzy logic approach. As stated above, the research gap remains when it comes to using hesitant fuzzy sets in geometric modeling, as the fuzzy sets applied in previous research assign single membership degree. While traditional fuzzy sets help manage uncertainty, they may not fully capture the hesitation often found in real-life decisions [14].

Despite these advancements, a critical research gap remains: no existing geometric modeling framework has integrated hesitant fuzzy sets, which can capture multiple degrees of membership and better represent hesitancy in real-world data. Traditional fuzzy models fall short when decision-makers usually ignore the hesitancy, which is a common occurrence during data collection. Therefore, this study aims to address this gap by proposing a novel hesitant fuzzy B-spline curve (HFBsC) approximation model, which applies hesitant fuzzy set theory into the B-spline curve framework. The study will start will a comprehensive literature review, followed by the study of preliminaries of concept such as hesitant fuzzy set, hesitant fuzzy point and hesitant fuzzy relation. Then, the study will apply the concept of hesitant fuzzy point relation and hesitant fuzzy control point relation defined by previous research to construct a novel model, namely hesitant fuzzy B-spline curve (HFBsC) approximation model. A quintic HFBsC approximation model is constructed and visualized as an example to show and verify the properties of HFBsC. The paper concludes by summarizing key insights and offering recommendations for future research directions.

2. Preliminaries

In this section, fundamental concepts including hesitant fuzzy set (HFS), hesitant fuzzy fuzzy point (HFP) and hesitant fuzzy relation (HFR) will be discussed, along with the definition of hesitant fuzzy point relation (HFPR) and hesitant fuzzy control point relation (HFCPR) to construct the model.

In 2010, Torra introduced a novel concept, namely hesitant fuzzy set theory, which aims to represent the uncertainty and ambiguity of each element in the fuzzy set. Instead of assigning single membership degree to elements, hesitant fuzzy set provides multiple membership representation for each element, which is more common in real-life situations. Definition 1 below shows the definition of hesitant fuzzy set.

Definition 1. [14,16,17] Let X be a fixed set. A hesitant fuzzy set (HFS) on X is in terms of a function applying to X and returns a subset of $[0,1]$. Mathematically, a HFS can be described as follows:

$$A = \{\langle x, h_A(x)\rangle | x \in X\} \tag{1}$$

where $h_A(x)$ is a set of some values in $[0,1]$, denotes the possible membership degrees of the element $x \in X$ to the set A. In this case, $h = h_A(x)$ is defined as a hesitant fuzzy element (HFE), while Θ represents the set of all HFEs. As the HFEs can consist of more than one membership degrees, some special HFEs for $x \in X$ are given as follows [17]:

i. Empty set: $h = \{0\}$, denoted as O^* as simplification.
ii. Full set: $h = \{1\}$, denoted as I^*.
iii. Complete ignorance (all are possible): $h = [0,1] \triangleq U^*$.
iv. Nonsense set: $h = \emptyset^*$.

To implement the concept of hesitant fuzzy set in B-spline curve modeling, the concepts of hesitant fuzzy point and hesitant fuzzy relation are necessary to represent the fuzzy behavior of control points and their relations. Definition 2 and Definition 3 below define the concept of hesitant fuzzy point and hesitant fuzzy relation.

Definition 2. [18] Let $Y \subseteq X$ and $A = \{\langle x, h_A(x)\rangle | x \in X\}$ such that $h_A(x) \subseteq [0,1]$. Then, $A_Y \in X$ is defined as follows:

$$A_Y(x) = \begin{cases} A, & for\ x \in Y \\ \{0\}, & x \notin Y \end{cases} \tag{2}$$

If Y is a singleton, namely $\{y\}$, then $A_{\{y\}}$ is called a hesitant fuzzy point (HFP) denoted by y_A.

Definition 3. [19,20] Suppose X and Y are universal sets. A hesitant fuzzy subset $\mathcal{R}$ of $X \times Y$ is defined as a hesitant fuzzy relation (HFR) from X to Y, that is,

$$\mathcal{R} = \{\langle (x,y), h_{\mathcal{R}}(x,y)\rangle | (x,y) \in X \times Y\} \tag{3}$$

For all $(x,y) \in X \times Y$, $h_{\mathcal{R}}(x,y)$ is a set of values in $[0,1]$, which are the possible membership degrees of the relations for each element x and y.

2.1 Hesitant Fuzzy Point Relation

Through the fundamental definitions in Definition 1, 2 and 3, Pa et al. have defined the hesitant fuzzy point relation (HFPR) to represent the fuzzy relation between each fuzzy element x and each fuzzy element y [21].

Definition 4. [21] Suppose X and Y are universal sets. Let $P \subseteq X$, $Q \subseteq Y$, $A = \{\langle x, h_A(x)\rangle | x \in X\}$ and $B = \{\langle y, h_B(x)\rangle | y \in Y\}$. Then by Definition 1, $h_A(x)$ and $h_B(x)$ are two sets of some possible values in $[0,1]$. Let P and Q are two HFPs, then by Definition 2, $A_{P=\{p\}} = \{\langle x, h_A(x)\rangle | x \in P\}$ and $B_{Q=\{q\}} = \{\langle y, h_B(x)\rangle | y \in Q\}$, where $\{p\}$ and $\{q\}$ are two singletons for sets P and Q respectively. A hesitant fuzzy subset $\mathcal{R}^*$ of $P \times Q$ is defined as a hesitant fuzzy point relation (HFPR) from X to Y, as follows:

$$\mathcal{R}^* = \{\langle (x_i, y_j), h_{R^*}(x_i, y_j)\rangle | x_i \in P_i, y_j \in Q_j, (x_i, y_j) \in P_i \times Q_j\} \quad (4)$$

For all $(x_i, y_j) \in P \times Q$, $h_{\mathcal{R}^*}(x_i, y_j) \in h_A(x_i) \times h_B(y_j)$ is the set of values in [0,1], which denotes the possible membership degrees of the HFPR for x_i and y_j.

2.2 Hesitant Fuzzy Control Point Relation

Control points are a set of points to control the shape and behavior of a B-spline curve. Changing in the position of control points will result in changing of the approximation of geometry. As fuzzy set is applied in this research, Pa et al. has defined the hesitant fuzzy control point relation (HFCPR) to represent the hesitancy of each hesitant fuzzy control point.

Definition 5. [22] Suppose P represents the control points of a Bézier curve such that

$$P = \{P_0, P_1, P_2, \dots, P_n\} \quad (5)$$

where P is a set of $n+1$ points denoting the coordinates of each control point. Let $\mathcal{R}^*$ be a HFPR, then the hesitant fuzzy control points relation (HFCPR) is defined as a relation of $(n+1)$ points with different number of choices as follows:

$$\mathcal{R}^*_{cp} = \left\{\langle \left(\mathcal{P}^{\mathcal{H}}_{0,j_0}, \mathcal{P}^{\mathcal{H}}_{1,j_1}, \dots, \mathcal{P}^{\mathcal{H}}_{n,j_n}\right), h_{R^*}\left(\mathcal{P}^{\mathcal{H}}_{0,j_0}, \mathcal{P}^{\mathcal{H}}_{1,j_1}, \dots, \mathcal{P}^{\mathcal{H}}_{n,j_n}\right)\rangle | \mathcal{P}^{\mathcal{H}}_{i,j_i} \in \mathcal{P}^{\mathcal{H}}_{i,M_i}, \left(\mathcal{P}^{\mathcal{H}}_{0,j_0}, \mathcal{P}^{\mathcal{H}}_{1,j_1}, \dots, \mathcal{P}^{\mathcal{H}}_{n,j_n}\right) \in \mathcal{P}^{\mathcal{H}}_{0,M_0} \times \mathcal{P}^{\mathcal{H}}_{1,M_1} \times \dots \times \mathcal{P}^{\mathcal{H}}_{n,M_n}\right\} \quad (6)$$

where $j_i \in M_i$, $M_i = \{0,1,2,\dots,m_i\}$ and $\mathcal{P}^{\mathcal{H}}_{i,M_i}$ are the hesitant fuzzy control points (HFCPs) at i-th term. Noted that $m_i = m_j$ for $i \neq j$ is possible but not compulsory. For each $\mathcal{P}^{\mathcal{H}}_{i,M_i}$, there are m_i number of possible memberships, therefore, the relation returns in a set of possible HFCPs. The HFCP at i-th term are defined as follows:

$$\begin{aligned}
\mathcal{P}^{\mathcal{H}}_{0,j_0\in M_0} &= \left\{\mathcal{P}^{\mathcal{H}}_{0,0}, \mathcal{P}^{\mathcal{H}}_{0,1}, \dots, \mathcal{P}^{\mathcal{H}}_{0,m_0}\right\} \\
\mathcal{P}^{\mathcal{H}}_{1,j_1\in M_1} &= \left\{\mathcal{P}^{\mathcal{H}}_{1,0}, \mathcal{P}^{\mathcal{H}}_{1,1}, \dots, \mathcal{P}^{\mathcal{H}}_{1,m_1}\right\} \\
\mathcal{P}^{\mathcal{H}}_{2,j_2\in M_2} &= \left\{\mathcal{P}^{\mathcal{H}}_{2,0}, \mathcal{P}^{\mathcal{H}}_{2,1}, \dots, \mathcal{P}^{\mathcal{H}}_{2,m_2}\right\} \\
&\vdots \\
\mathcal{P}^{\mathcal{H}}_{n,j_n\in M_n} &= \left\{\mathcal{P}^{\mathcal{H}}_{n,0}, \mathcal{P}^{\mathcal{H}}_{n,1}, \dots, \mathcal{P}^{\mathcal{H}}_{n,m_n}\right\}
\end{aligned} \quad (7)$$

In this case, HFCP is a set of $n+1$ control points, where each control points have m_i possibilities based on different hesitant fuzzy membership degrees assigned. Therefore, there are a variety of possible HFCPRs, that is, a variety of possible HFCPs produced. The total number of HFCPRs, namely $N_{\mathcal{R}^*_{cp}}$ is denoted by equation below:

$$N_{\mathcal{R}^*_{cp}} = \prod_{i=0}^{n} m_i \quad (8)$$

The HFCP can be redefined as

$$\mathcal{P}^{\mathcal{H}}_k = \left\{\left\{\mathcal{P}^{\mathcal{H}}_{0,j_0}, \mathcal{P}^{\mathcal{H}}_{1,j_1}, \mathcal{P}^{\mathcal{H}}_{2,j_2}, \dots, \mathcal{P}^{\mathcal{H}}_{n,j_n}\right\} | j_i \in M_i, M_i = \{0,1,2,\dots,m_i\}, \mathcal{P}^{\mathcal{H}}_{i,j_i} \in \mathcal{P}^{\mathcal{H}}_{i,M_i}\right\} \quad (9)$$

for $N_{\mathcal{R}_{cp}^*}$ possibilities, where $k = 1,2,, \dots N_{\mathcal{R}_{cp}^*}$. Therefore, the set of all HFCPs is described as follows:

$$\boldsymbol{\mathcal{P}^{\mathcal{H}}} = \left\{\mathcal{P}_k^{\mathcal{H}} | k = 1,2, \dots, N_{\mathcal{R}_{cp}^*}\right\} \tag{10}$$

3. Hesitant Fuzzy B-spline Approximation Model

The traditional Bézier curve uses Bernstein polynomials, whereas a B-spline curve is constructed from B-spline basis functions, allowing more control by manipulating the degree of blending functions. By using the definition of hesitant fuzzy control point (HFCP) and the concept of hesitant fuzzy control point relation (HFCPR), the hesitant fuzzy B-spline curve (HFBsC) approximation model is extended as below.

Definition 6: Let $\boldsymbol{\mathcal{P}^{\mathcal{H}}} = \left\{\mathcal{P}_k^{\mathcal{H}} | k = 1,2, \dots, N_{\mathcal{R}_{cp}^*}\right\}$ be the set of all HFCPs such that $\mathcal{P}_k^{\mathcal{H}} = \left\{\{\mathcal{P}_{0,j_0}^{\mathcal{H}}, \mathcal{P}_{1,j_1}^{\mathcal{H}}, \mathcal{P}_{2,j_2}^{\mathcal{H}}, \dots, \mathcal{P}_{n,j_n}^{\mathcal{H}}\} | j_i \in M_i, M_i = \{0,1,2, \dots, m_i\}, \mathcal{P}_{i,j_i}^{\mathcal{H}} \in \mathcal{P}_{i,M_i}^{\mathcal{H}}\right\}$ is the possible HFCPs. Let $BsC(t)$ as the curve position vector of a B-spline curve. A HFBsC approximation model, namely $\boldsymbol{BsC^{\mathcal{H}}}(t)$ with order k and degree $(k-1)$ is defined as follows:

$$\boldsymbol{BsC^{\mathcal{H}}}(t) = \sum_{i=0}^{n} \boldsymbol{\mathcal{P}^{\mathcal{H}}} \cdot N_{i,k}(t) \qquad t_{min} \leq t \leq t_{max} \tag{11}$$

where t is the parameter and the B-spline basis, $N_{i,k}(t)$ is defined by the Cox-de-Boor recursion formula as follows:

$$N_{i,k}(t) = \begin{cases} 1, & \text{if } x_i \leq t < x_{i+1} \\ 0, & \text{otherwise.} \end{cases} \tag{12}$$

and

$$N_{i,k}(t) = \frac{(t-x_i)N_{i,k-1}(t)}{x_{i+k-1}-x_i} + \frac{(x_{i+k}-t)N_{i+1,k-1}(t)}{x_{i+k}-x_{i+1}} \tag{13}$$

where the value of x_i are the elements of a knot vector satisfying the relation $x_i \leq x_{i+1}$. Restrictions $(0)^0 \equiv 1$ and $0! \equiv 1$ are considered in this case. As there are multiple sets of HFCPs applied as input of this model, therefore, the HFBsC approximation model will construct the same number of HFBsCs as its output. The number of HFBsCs generated are the same with the number of HFCPRs, which denoted by Eq. (8). In short, a $\boldsymbol{BsC^{\mathcal{H}}}(t)$ is the set of all possible HFBsCs, $BsC_a^{\mathcal{H}}(t)$ such that

$$\boldsymbol{BsC^{\mathcal{H}}}(t) = \left\{BC_a^{\mathcal{H}}(t) = \sum_{i=0}^{n} \mathcal{P}_a^{\mathcal{H}} \cdot N_{i,k}(t) \, | a = 1,2,3, \dots, N_{\mathcal{R}_{cp}^*}\right\} \tag{14}$$

In that the case, $N_{\mathcal{R}_{cp}^*}$number of HFBsCs will be generated through the model.

n summary, the Hesitant Fuzzy B-spline Curve (HFBsC) approximation model utilizes multiple sets of Hesitant Fuzzy Control Points (HFCPs) as input, each representing different combinations of control point memberships due to hesitancy. As a result, the model produces a corresponding set of possible B-spline curves, where each output curve reflects a unique configuration of the control points. This enables the modeling of uncertainty and hesitation in geometric approximation. A numerical example illustrating the construction and behavior of the HFBsC model will be provided in Section 4 for further clarification and discussion.

3.1 Properties of Hesitant Fuzzy B-spline Curve

As the hesitant fuzzy B-spline curve model applies the B-spline basis as blending function, therefore the properties of HFBsC are listed as follows:

i. The basis functions of HFBsC are real.
ii. The sum of the B-spline basis functions for any parameter value is 1.
iii. Each basis function is positive or zero for all parameter values.
iv. The maximum order of the HFBsC equals the number of HFCP vertices. The maximum degree is one less.
v. The HFBsC exhibits variation-diminishing property.
vi. The HFBsC generally follows the shape of the control polygon formed by HFCP.
vii. The HFBsC lies within the convex hull of its control polygon.

4. Visualization of Quintic Hesitant Fuzzy B-spline Curve Approximation Model

In this section, a quintic (5-th order) hesitant fuzzy B-spline curve (HFBsC) approximation model will be constructed. Suppose there are 5 control points to generate the curve using the model stated. Let the (x, y) coordinates for the control points remain unchanged, while assume the membership degrees to represent the z-coordinates. Three of the points have 2 membership degrees assigned, therefore hesitancy occurs, since there are multiple sets of control points to construct the curves. Table 1 below shows the coordinates of the hesitant fuzzy control points.

Table 1
Coordinates of control points

(x, y)	z-coordinates	
	z_1	z_2
(0,0)	0	-
(0.3,0.3)	0.8	0.3
(0.5,0.5)	0.6	-
(0.8, 0.9)	1	0.8
(0.9,1)	0.5	0.1

By Eq. (8), the total number of HFCPs can be calculated as below.

$$N_{\mathcal{R}_{cp}^*} = \prod_{i=0}^{4} m_i = 2 \cdot 2 \cdot 2 = 8$$

$$\boldsymbol{\mathcal{P}^{\mathcal{H}}} = \left\{\mathcal{P}_1^{\mathcal{H}}, \mathcal{P}_2^{\mathcal{H}}, \mathcal{P}_3^{\mathcal{H}}, \mathcal{P}_4^{\mathcal{H}}, \mathcal{P}_5^{\mathcal{H}}\right\}$$

$$\boldsymbol{BsC^{\mathcal{H}}}(t) = \left\{BsC_1^{\mathcal{H}}(t), BsC_2^{\mathcal{H}}(t), BsC_3^{\mathcal{H}}(t), BsC_4^{\mathcal{H}}(t), BsC_5^{\mathcal{H}}(t)\right\}$$

Therefore, the set of HFCPRs and HFBsCs are stated as above. There are 8 sets of $\boldsymbol{\mathcal{P}^{\mathcal{H}}}$ and $\boldsymbol{BsC^{\mathcal{H}}}(t)$ as well. Figure 1 below shows the output of HFBsC approximation model on the same axes, while Figure 2 (a) to Figure 2 (h) shows the visualization of each HFBsCs separately by each respective HFCPs.

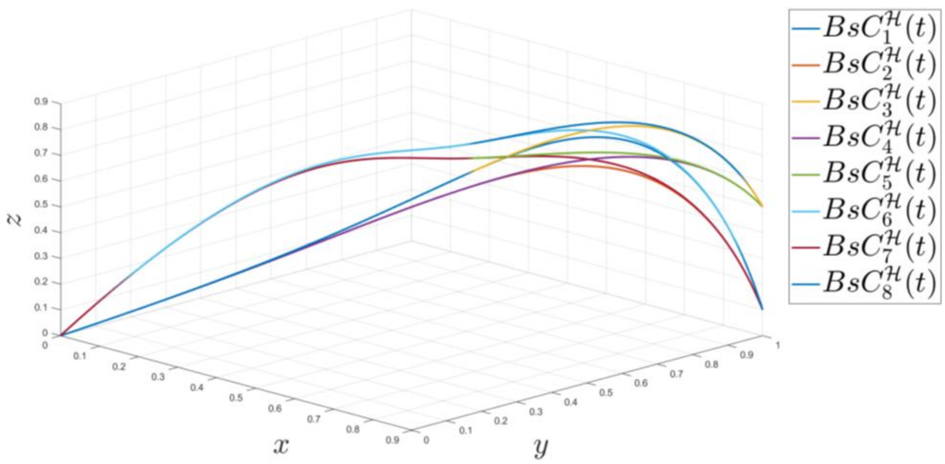
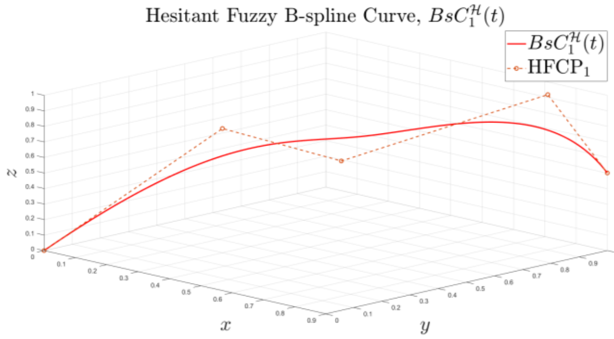
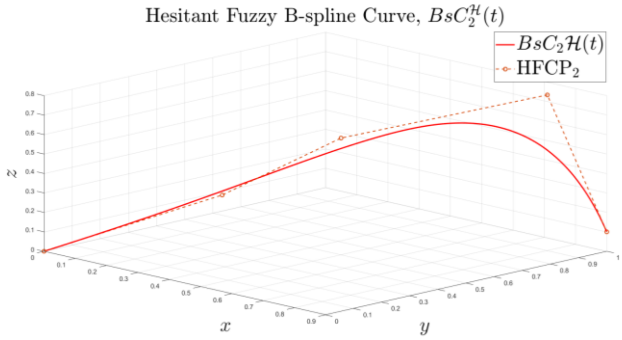


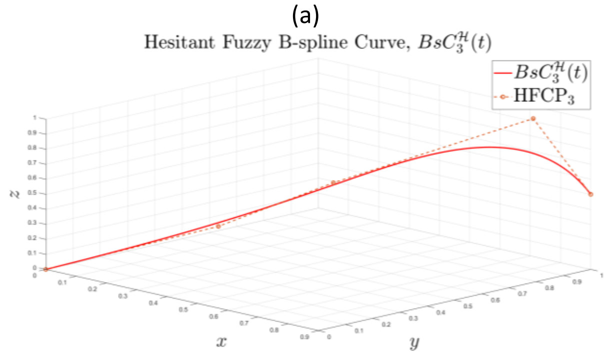
Fig. 1. Hesitant fuzzy B-spline Curves, $\boldsymbol{BsC^{\mathcal{H}}}(t)$



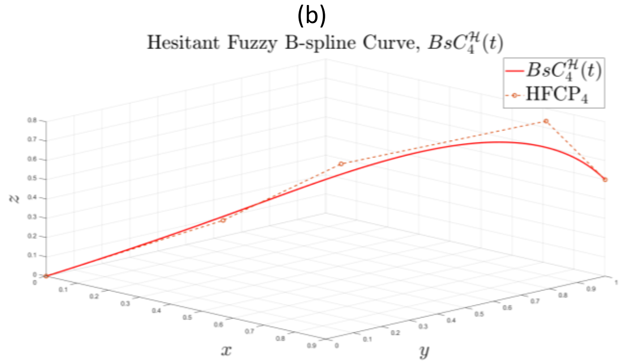
(a)



(b)



(c)



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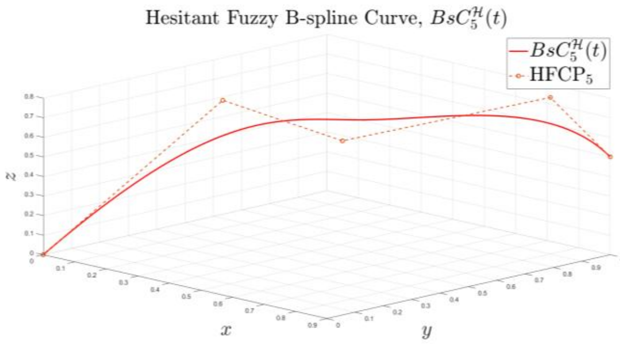
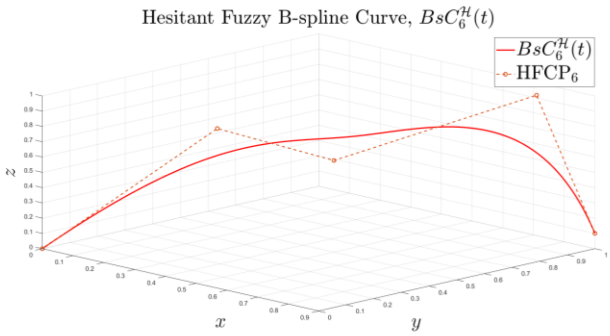
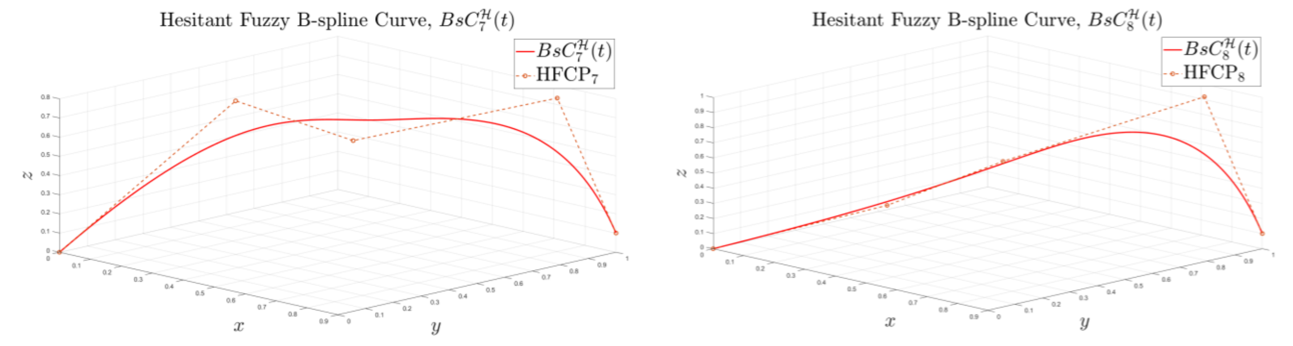


(e)



(f)



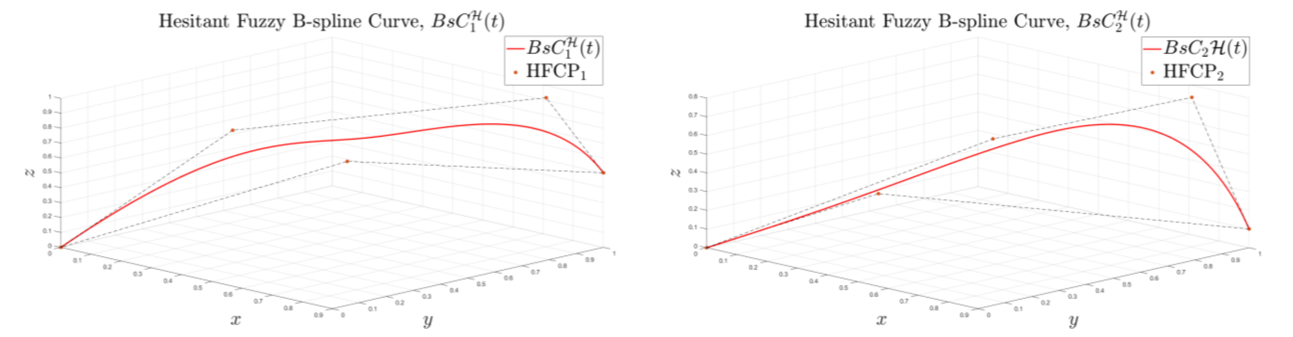
(g) (h)

Fig. 2. (a) HFBsC, $BsC_1^{\mathcal{H}}(t)$, (b) HFBsC, $BsC_2^{\mathcal{H}}(t)$, (c) HFBsC, $BsC_3^{\mathcal{H}}(t)$, (d) HFBsC, $BsC_4^{\mathcal{H}}(t)$, (e) HFBsC, $BsC_5^{\mathcal{H}}(t)$, (f) HFBsC, $BsC_6^{\mathcal{H}}(t)$, (g) HFBsC, $BsC_7^{\mathcal{H}}(t)$, (h) HFBsC, $BsC_8^{\mathcal{H}}(t)$

4. Discussion

Through Figure 1, it can be noticed that the HFBsCs visualized are different due to the hesitancy of data. Figure 2(a) to Figure 2(h) shows the HFBsCs generated by their respective HFCPs. It can be observed that all quintic HFBsCs are controlled by the control polygon (formed by the HFCPs). Besides, all HFBsCs are bound by their largest convex hull of the control polygon. Moreover, the first point and the last point of the HFCPs coincide with the two ends of the HFBsCs for each figure. These observations show that all HFBsCs achieve the properties to be HFBsC. Figure 3(a) to Figure 3(h) below shows the observation.

This study introduces a novel approach to geometric modeling, particularly to B-spline curve with hesitant fuzzy set. The construction of model namely hesitant fuzzy B-spline curve (HFBsC) approximation model provides a technique to address the uncertainty, especially the hesitancy of data collection which is always ignoring some opinions in real-life contexts. The mathematical representation of HFBsC approximation model is defined, along with the HFCPs and HFCPRs, which will be beneficial to the study of handling uncertainty and provide future insights into an interpolation model, which is more applicable in real-world scenarios.



(a) (b)

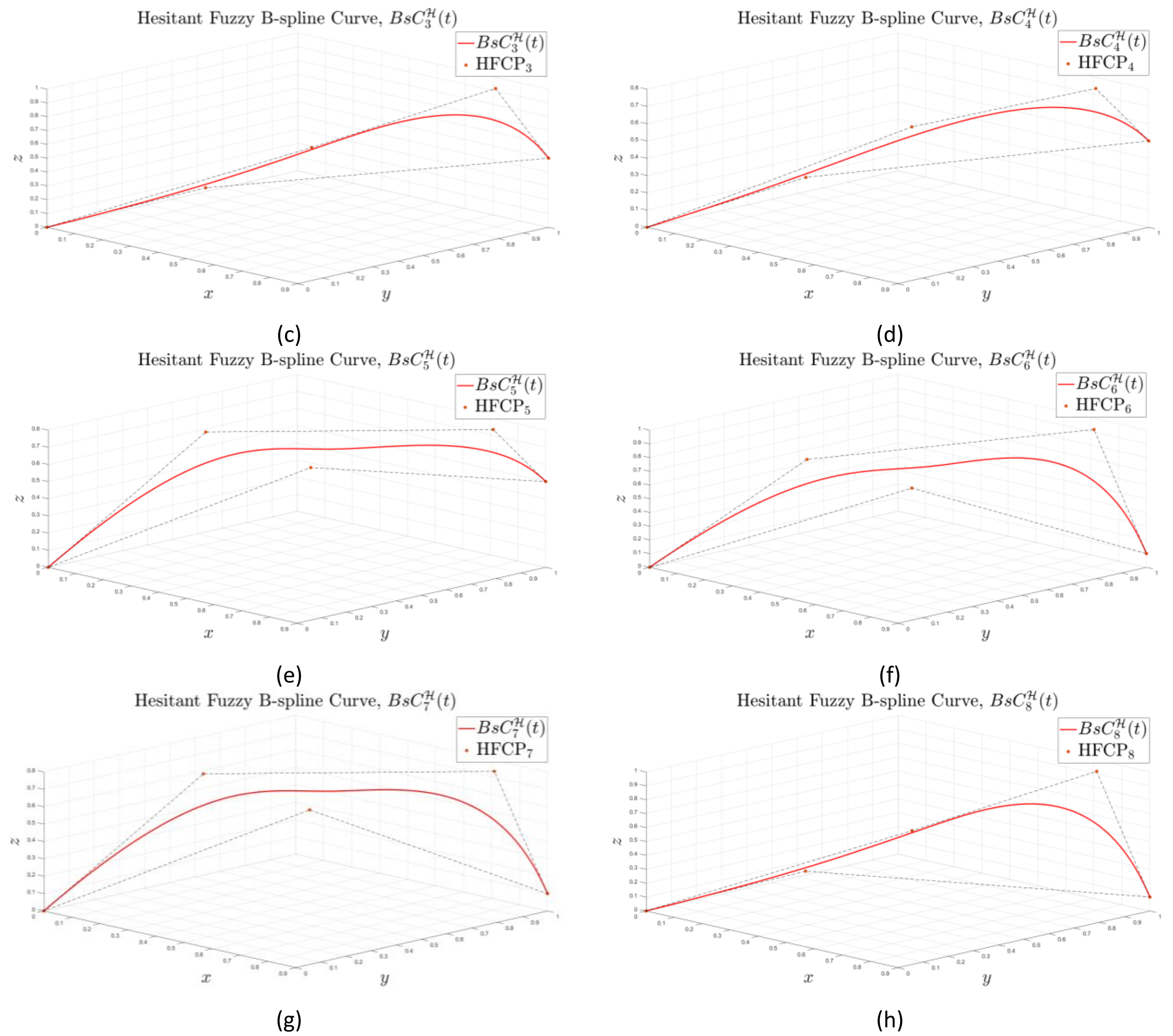


Fig. 3. (a) HFBsC, $BsC_1^{\mathcal{H}}(t)$, (b) HFBsC, $BsC_2^{\mathcal{H}}(t)$, (c) HFBsC, $BsC_3^{\mathcal{H}}(t)$, (d) HFBsC, $BsC_4^{\mathcal{H}}(t)$, (e) HFBsC, $BsC_5^{\mathcal{H}}(t)$, (f) HFBsC, $BsC_6^{\mathcal{H}}(t)$, (g) HFBsC, $BsC_7^{\mathcal{H}}(t)$, (h) HFBsC, $BsC_8^{\mathcal{H}}(t)$ with control polygon (maximum convex hull)

5. Conclusions

To conclude, the study constructs a novel model, namely hesitant fuzzy B-spline curve (HFBsC) approximation model to address the hesitancy and uncertainty of data. The numerical example of quintic HFBsC approximation model is visualized to show that the output of the model satisfies the properties of hesitant fuzzy B-spline curve. The application of hesitant fuzzy set allows the study to consider every opinion on the data during the visualization instead of neglecting some of them. Nevertheless, the HFBsC approximation model can only provide visualization of hesitancy, but it is unable to make a produce a crisp result. Therefore, an appropriate justification method is necessary as future research.

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References

[1] Bézier, Pierre. "Numerical control-mathematics and applications." *Translated by AR Forrest* (1972).
[2] De Boor, Carl, and Carl De Boor. *A practical guide to splines*. Vol. 27. New York: springer, 1978. https://doi.org/10.1007/978-1-4612-6333-3
[3] Piegl, Les, and Wayne Tiller. *The NURBS book*. Springer Science & Business Media, 2012.
[4] Rogers, David F. *An introduction to NURBS: with historical perspective*. Elsevier, 2000.
[5] Foley, James D. *Computer graphics: principles and practice*. Vol. 12110. Addison-Wesley Professional, 1996.
[6] Perwass, Christian, Herbert Edelsbrunner, Leif Kobbelt, and Konrad Polthier. *Geometric algebra with applications in engineering*. Vol. 4. Berlin: Springer, 2009.
[7] Salomon, David. *Computer graphics and geometric modeling*. Springer Science & Business Media, 2012.
[8] Zadeh, Lotfi Asker. "Fuzzy sets." *Information and control* 8, no. 3 (1965): 338-353. https://doi.org/10.1016/S0019-9958(65)90241-X
[9] Tan, Woei Wan, and Teck Wee Chua. "Uncertain rule-based fuzzy logic systems: introduction and new directions (Mendel, JM; 2001)[book review]." *IEEE Computational intelligence magazine* 2, no. 1 (2007): 72-73. https://doi.org/10.1109/MCI.2007.357196
[10] Atanassov, Krassimir. "Intuitionistic fuzzy sets." *International journal bioautomation* 20 (2016): 1. https://doi.org/10.1016/S0165-0114(86)80034-3
[11] Klement, Erich Peter, Radko Mesiar, and Endre Pap. *Triangular norms*. Vol. 8. Springer Science & Business Media, 2013.
[12] Zadeh, Lotfi Asker. "The concept of a linguistic variable and its application to approximate reasoning—I." *Information sciences* 8, no. 3 (1975): 199-249. https://doi.org/10.1016/0020-0255(75)90036-5
[13] Liu, Yanhui, and Yong Liu. "Picture Fuzzy Sets." *Journal of Computer Science and Technology* 22, no. 5 (2007): 778–785.
[14] Torra, Vicenç, and Yasuo Narukawa. "On hesitant fuzzy sets and decision." In *2009 IEEE international conference on fuzzy systems*, pp. 1378-1382. IEEE, 2009. https://doi.org/10.1109/FUZZY.2009.5276884
[15] Anile, Angelo M., Salvatore Deodato, and Giovanni Privitera. "Implementing fuzzy arithmetic." *Fuzzy sets and Systems* 72, no. 2 (1995): 239-250. https://doi.org/10.1016/0165-0114(94)00355-B
[16] Torra, Vicenç. "Hesitant fuzzy sets." *International journal of intelligent systems* 25, no. 6 (2010): 529-539. https://doi.org/10.1002/int.20418
[17] Xia, Mingjun, and Zeshui Xu. "Hesitant Fuzzy Information Aggregation in Decision Making." *International Journal of Approximate Reasoning* 52, no. 3 (2011): 395–407. https://doi.org/10.1016/j.ijar.2010.10.002
[18] Deepak, D., and Sunil Jacob John. "Homomorphisms of hesitant fuzzy subgroups." *International Journal of scientific and engineering research* 5, no. 9 (2014): 9-14.
[19] Deepak, D., and Sunil Jacob John. "Hesitant fuzzy rough sets through hesitant fuzzy relations." *Annals of Fuzzy mathematics and Informatics* 8, no. 1 (2014): 33-46.
[20] Yang, Xibei, Xiaoning Song, Yunsong Qi, and Jingyu Yang. "Constructive and axiomatic approaches to hesitant fuzzy rough set." *Soft Computing* 18 (2014): 1067-1077. https://doi.org/10.1007/s00500-013-1116-2
[21] Wei, Jin Pa, Mohammad Izat Emir Zulkifly, and T. K. Ahmad Khairuddin. "Biquartic Hesitant Fuzzy Bézier Surface Approximation Model with Its Visualization." *Semarak International Journal of Fundamental and Applied Mathematics* 3, no. 1 (2024): 1–10. https://doi.org/10.37934/sijfam.3.1.110
[22] Pa, Wei Jin, Mohammad Izat Emir Zulkifly, and TK Ahmad Khairuddin. "7th order hesitant fuzzy Bézier curve interpolation model with simple average justification of hesitancy." In *AIP Conference Proceedings*, vol. 3244, no. 1, p. 020059. AIP Publishing LLC, 2024. https://doi.org/10.1063/5.0242320