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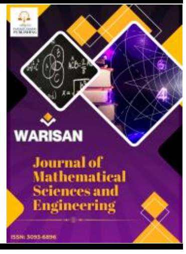

A Comparative Study of First Order Polarization in Spherical Structures: Exploring Numerical Approaches

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ABSTRACT

The first-order polarization tensor (PT) is the basic term of the generalized polarization tensor (GPT), which represents the asymptotic series in the form of an integral equation. It is crucial in engineering applications, such as material characterization and electromagnetic modeling. Many approaches have been implemented to enhance computational efficiency and accuracy in the numerical computation of the first-order polarization tensor. Among these methods, two have received the attention of the researchers, but their comparative effectiveness remains uncertain. Hence, this study aims to analyse and compare these two methods in approximating the first-order polarization tensor for a specific object inclusion, the sphere. The methodology involves numerical simulations where the performance of both methods is observed. From the results obtained, both methods satisfy the theoretical statement that as the number of mesh discretization increases, the accuracy of the numerical simulations improves. Two types of software are used in this study: Netgen Mesh Generator and MATLAB, which assist in the pre-processing, processing, and post-processing stages of numerical computation. The simulation and numerical examples will help verify the comparative analysis of both methods.

1. Introduction

First order polarization tensor (PT) is a mathematical tool that is widely used to characterize objects, as it has the ability to represent the shape of an object represented in 3×3 matrix in xyz direction. Various applications, such as metal detection, use PT for which PT has the ability to aid in analyzing as well as characterizing the electromagnetic signature of any metallic object inclusions. Hence, accurate numerical computation of first order PT is essential in enhancing the precision of the classification as well as identification of the object inclusions which is crucial in application such as security screening, exploration of archeological object as well as detection of unexploded ordnance [1]. Not even that, since the information about an object properties, conductivity as well as shape of the given object can be interpreted by PT, for application like eddy current response, PT influence directly the eddy current response and scatter field measurements. The properties of PT make it an

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important tool that can be used in the application of electromagnetic imaging and inverse problem. Other application that can be seen from PT is in the system of metal detection where most of the system relies on the inverse problem related to propagation of electromagnetic wave. A metal can be detected by observing the response of the materials to an experienced magnetic field. This response can be determined by the properties, position of the magnetic field as well as the shape of the object inclusion [2]. Due to this, an accurate numerical computation of PT must be obtained to ensure that we have a reliable result, which has yet to be fully addressed. To date, there were many computational techniques has been implemented to solve the first order PT such as semi-algebraic method, boundary element method (BEM), finite element method (FEM) and many others [3-7,14-16]. However, when these method is implemented in the calculation of PT, they often come with a higher computation cost since the PT integral consist of singularity which will lead to a complex computation of the integral. To achieve an accurate numerical computation of PT, a finer mesh need to be used, where in this case, it will increase the computational complexity and this will lead to inefficient for real time application such as metal detection for which we know, the application need a faster result. This can be more challenging if the object inclusion is irregular in it shapes and highly detailed object inclusion [6]. Moreover, numerical integration techniques may suffer from instability when applied to objects with intricate boundary conditions, leading to potential inaccuracies in the polarization tensor estimation.

Other numerical techniques that has been explored by researchers are multivariate polynomial interpolation, for which this method can improve both the accuracy and computational efficiency of PT computation. This is due to its ability to approximate the values of the 9 tensor components based on a finite sample set of data points [9,10]. In a study conducted in a previous study [11], the multivariate polynomial interpolation was used, where the researchers constructed the interpolating polynomial function to estimate the first order PT and the polynomial function is represented as:

$$P_i(x,y) = C_0 + C_1 y + C_2 x + C_3 y^2 + C_4 xy + C_5 x^2 + \cdots + C_i x^m y^n, \qquad \text{for } i = 1,2,3,\cdots k \tag{1}$$

where $P_i(x,y)$ is the polynomial function with constant coefficient, $C_0, C_1, C_2, \cdots, C_i$.

On the other hand, as aforementioned, a finite element based technique is also used by researcher to compute first order PT where this technique can improve the accuracy of the numerical solution by ensuring higher order approximations, and is widely used in the application of electromagnetic distribution. The ability to deal with complex boundary conditions makes this numerical technique more effective than other numerical techniques. In the research conducted by Sukri *et al.,* [10], it is observed that, as a quadratic element was used in the computation of PT, the result shows higher accuracy when it is compared with its analytical solution. Despite the advantages offered by both multivariate interpolation and quadratic element integration, a direct comparison of their effectiveness in computing the first-order polarization tensor has not been extensively explored. While both methods offer potential improvements over traditional numerical integration techniques, their relative performance in terms of computational efficiency, accuracy, and applicability to real-world metal detection scenarios, for example, remains an open question. This study aims to bridge this gap by conducting a comparative analysis of these two computational techniques, assessing their strengths and limitations in different problem domains.

To achieve a comprehensive evaluation, this study will consider a benchmark case which includes simple geometric shapes such as a sphere. By comparing the accuracy, computational time, and numerical stability of multivariate interpolation and quadratic element integration, we aim to provide insights that can guide the selection of the most appropriate method for different metal detection scenarios. One parameter that is considered in this paper is the conductivity of the object inclusion.

This parameter is chosen as the primary parameter because, for example, in metal detection applications, this parameter will determine the electromagnetic response of the object inclusion and can influence the representation of the matrix of first order PT. However, we cannot deny that other parameters such as the shape of the object, permeability and frequency of the applied electromagnetic fields also play an important role in the PT representation. By considering this parameter, we will observe and compare the behavior of both methods with respect to increasing values of conductivity of the object inclusion. The rest of this paper is structured as follows: Section 2 presents the mathematical formulation of the first-order polarization tensor, providing a theoretical background for both computation methods. Section 3 outlines the methodologies employed for multivariate interpolation and quadratic element integration, detailing their implementation and computational considerations. Section 4 discusses the comparative results obtained from numerical experiments, analyzing the advantages and limitations of each technique. Finally, Section 5 concludes with key findings and potential future research directions, highlighting areas where further improvements in FPT computation can be made. Through this comparative study, we aim to contribute to the ongoing advancements in metal detection technology and electromagnetic modeling. By evaluating the trade-offs between accuracy and computational efficiency, we hope to provide a clear framework for selecting the most appropriate numerical technique for different applications. Our findings may serve as a valuable reference for researchers and engineers seeking to optimize polarization tensor computations in practical scenarios, ultimately leading to enhanced detection capabilities and more efficient electromagnetic analysis techniques.

2. Mathematical Framework of First Order Polarization Tensor

2.1 Main Integral of PT

In this subsection, the main integral for formulation of PT which consist of three main equations is defined. These three equations are usually used in the application of effective medium theories, inverse problems and electromagnetic scattering where it characterizes the response of the object inclusion. The first order PT integral can be represented by a rank two tensor which comes from the generalized form of PT (GPT) for which the leading term is the first order PT. It is represented in integral equation for domain Ω as in Eq. (2)

$$M(k,\Omega) = \int_{\partial\Omega} Y^j \Phi_i(Y) d\sigma(Y), \qquad Y \in \partial\Omega \tag{2}$$

where k is the conductivity of the object inclusion Ω, Y is the element of the object domain while $\Phi_i(Y)$ is defined as linear system of equation as in Eq. (3)

$$\Phi_i(Y) = \left(AI - K_B^*\right)^{-1}\left(\hat{V}_X\right), \qquad Y \in \partial\Omega. \tag{3}$$

Eq. (3) is in the form of linear system of equation where, $A = \left(k+1\right)/2\left(k-2\right)$ while $\hat{V}_X$ is the unit normal vector of element X represented as,

$$\hat{V}_X = \left(v_X \cdot \nabla X^i\right)\left(Y\right). \tag{4}$$

K_B^* is in the form of Cauchy principal integral (PV) as in Eq. (5) where $\Phi(X) \in L^2(\partial\Omega)$ and $L^2(\partial\Omega)$ is integrable function on domain Ω.

$$K_B^*\Phi_i(X_i) = \frac{1}{4\pi} PV \int_{\partial\Omega} \frac{\langle X_i - Y_j, v_{X_i} \rangle}{|X_i - Y_j|^3} \Phi(Y_j) d\sigma(Y_j). \tag{5}$$

From Eq. (5), $\langle X_i - Y_j \rangle$ is in the difference between the elements in domain Ω. Hence, in order to compute the first order PT, we need to solve for all three equations as in Eq. (2), (3) and (5) for which the only parameter that we will use in this study is only conductivity of the object inclusion. For next subsection, we are going to review the analytical solution of the first order PT for a sphere that has been described by previous researcher.

2.2 Explicit Formulae of the First Order PT

The analytical expression of the first order PT of Pólya-Szegö for disk and ellipse has been derived by Brühl *et al.,* [12] which is expressed in matrix of 2 by 2 as

$$M_A(k,\Omega) = (k-1)|\Omega| \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix}, \tag{6}$$

where the volume of the object (either disk or ellipse) is denoted as $|\Omega|$, while the first element and second element can be computed using the following equation with semi-axes a and b

$$M_1 = \frac{a+b}{a+kb} \text{ and } M_2 = \frac{a+b}{b+ka}. \tag{7}$$

In three dimensional case, the analytical expression has been derived and stated first for ellipsoid object inclusion. The semi-principal axes a, b and c of the ellipsoid with general equation $\frac{x^*}{a} + \frac{y^*}{b} + \frac{z^*}{c} = 1$, for $0 \le c \le b \le a$, its resulting expression for the analytical solution of the first order PT is

$$M_A(k,\Omega) = (k-1)|\Omega| \begin{bmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & M_3 \end{bmatrix}. \tag{8}$$

where M_1, M_2 and M_3 are defined by

$$M_1 = \frac{1}{\left(1 - \frac{b^* c^*}{\left(a^*\right)^2} \int_1^\infty \frac{1}{t^2 \left(t^2 - 1 + \left(\frac{b^*}{a^*}\right)^2\right)^{1/2} \left(t^2 - 1 + \left(\frac{b^*}{a^*}\right)^2\right)^{1/2}} dt\right) + k \frac{b^* c^*}{\left(a^*\right)^2} \int_1^\infty \frac{1}{t^2 \left(t^2 - 1 + \left(\frac{b^*}{a^*}\right)^2\right)^{1/2} \left(t^2 - 1 + \left(\frac{b^*}{a^*}\right)^2\right)^{1/2}} dt},$$

$$M_2 = \frac{1}{\left(1 - \frac{b^* c^*}{\left(a^*\right)^2} \int_1^\infty \frac{1}{t^2 \left(t^2 - 1 + \left(\frac{b^*}{a^*}\right)^2\right)^{3/2} \left(t^2 - 1 + \left(\frac{c^*}{a^*}\right)^2\right)^{1/2}} dt\right) + k \frac{b^* c^*}{\left(a^*\right)^2} \int_1^\infty \frac{1}{\left(t^2 - 1 + \left(\frac{b^*}{a^*}\right)^2\right)^{3/2} \left(t^2 - 1 + \left(\frac{c^*}{a^*}\right)^2\right)^{1/2}} dt},$$

$$M_3 = \frac{1}{\left(1 - \frac{b^* c^*}{\left(a^*\right)^2} \int_1^\infty \frac{1}{t^2 \left(t^2 - 1 + \left(\frac{b^*}{a^*}\right)^2\right)^{1/2} \left(t^2 - 1 + \left(\frac{b^*}{a^*}\right)^2\right)^{3/2}} dt\right) + k \frac{b^* c^*}{\left(a^*\right)^2} \int_1^\infty \frac{1}{\left(t^2 - 1 + \left(\frac{b^*}{a^*}\right)^2\right)^{1/2} \left(t^2 - 1 + \left(\frac{c^*}{a^*}\right)^2\right)^{3/2}} dt} \tag{9}$$

From the analytical solution of ellipsoid, in order to get an analytical solution of a sphere, which in this case our main concern of the study, we equate the semi-principal axes of an ellipsoid, $a^* = b^* = c^*$, this will make the object inclusion to become sphere and hence, the resulting element inside the analytical solution of PT, will become, $M_1 = M_2 = M_3 = 3/(2+k)$, yield to

$$M_A(k,\Omega) = (k-1)|\Omega| \begin{bmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & M_3 \end{bmatrix} \tag{10}$$

The derivation of the analytical solution for first order PT for sphere and ellipsoid can be used as a benchmark solution for other irregular object inclusion. Next subsection will present the mathematical modelling by using proposed methods which is the multivariate polynomial as well as quadratic element integration.

3. Mathematical Modelling

3.1 Discretization of the Object Inclusion: NG Solver

In this section, we will discuss the discretization of the object inclusion which is sphere by using a meshing tools. To date, there were many meshing tools that has been developed by a researcher such as in [17,18]. However, in this research, Netgen Mesh Generator or simply called NG Solver will be used for the discretization of the object inclusion. The latest version of NG Solver is version 6.2 where it is developed by Joachim Schöberl in 1997 [13]. The meshing strategies is available to the user such as, for simple geometries: structured grid, can be triangular meshes or tetrahedral meshes while for complex geometry: it uses unstructured mesh. The computation with the help of NG Solver is enhanced for which it allows the adaptive meshing technique that enables the refinement criteria for meshes with solution gradients or any other parameters. The interface of the meshing tools is demonstrated in Figure 1, while Figure 2 show the geometry setup interface without the refinement.

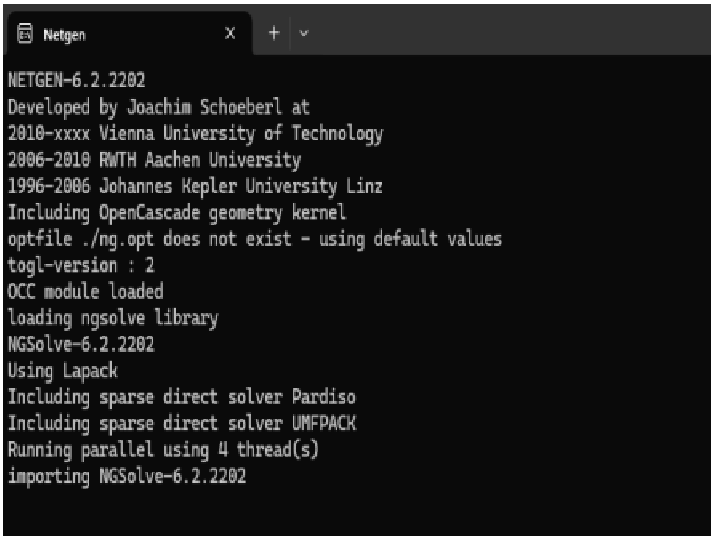

Fig. 1. The interface of the meshing tools, NG Solver by Joachim Schöberl

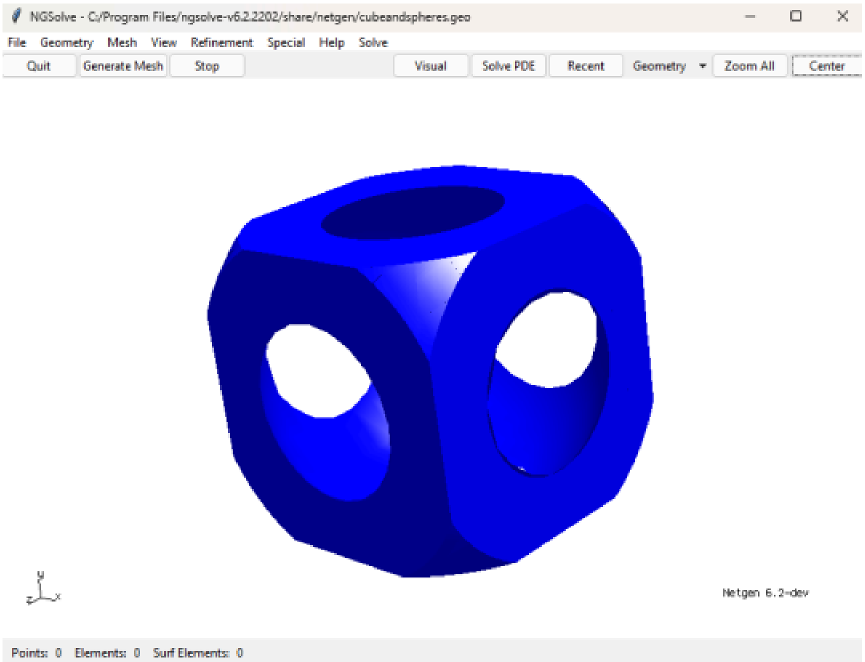

Fig. 2. The interface of the mesh generation for complex configuration by NG Solver

Since NG Solver provide a high quality in terms of mesh generation which is capable in dealing with structured geometries as well as unstructured meshes in complex geometries, the integration of NG Solver with the numerical computation will provide an efficient and reliable solution that can be used in real-life application.

3.2 Transformation of Main Integral of First Order PT: Quadratic Element Integration

This subsection will briefly recall the mathematical methodology of first order PT by using quadratic element integration by Sukri *et al.,* [10]. The mathematical development involved formulation of shape functions, Gaussian quadrature and the computation of the first order PT based on the three main diagonal as in Eq. (2), (3) and (5). The researcher transforms the integral of PT in (5) to become a summation function containing the Jacobian matrix, $J(\xi,\eta)$ in the form of xyz coordinates with shape functions, $x=\sum_{i=1}^{6} x_i N_i(\xi,\eta)$, $y=\sum_{i=1}^{6} y_i N_i(\xi,\eta)$ and $z=\sum_{i=1}^{6} z_i N_i(\xi,\eta)$ as

$$M(k,\Omega)=\Phi_i(Y)\sum_{l=1}^{6} w_l Y_l^j \sqrt{\left|J(\xi,\eta)^T \cdot J(\xi,\eta)\right|}, \tag{11}$$

where $J(\xi,\eta)=\begin{bmatrix} x_\xi & x_\eta, y_\xi & y_\eta, z_\xi & z_\eta \end{bmatrix}^T$ while the linear system of equation is written as

$$\Phi_i(Y)=\left(AI-K_B^*\right)^{-1}\left(v_X^{\alpha=1,2,3}\right). \tag{12}$$

The shape function, $N_i(\xi,\eta)$ are represented as

$$\begin{aligned}
N_1 &= 2\xi^2-\xi,\\
N_2 &= 2\eta^2-\eta,\\
N_3 &= 1-3\xi-3\eta+2\xi^2+2\eta\xi+2\eta^2,\\
N_4 &= 4\xi-4\xi^2-4\xi\eta,\\
N_5 &= 4\xi\eta,\\
N_6 &= -4\eta^2-4\xi\eta+4\eta.
\end{aligned} \tag{13}$$

The Cauchy Principal Value Integral, K_B^* approximation is also expressed as the summation containing the weighting function as well as the Jacobian matrix, $J(\xi,\eta)$ where Eq. (5) is transformed to

$$K_B^*\Phi_i(X_i)=\frac{1}{4\pi}\sum_{i,j=1}^{N}\frac{\left\langle X_i^{\alpha=1,2,3}-Y_j^{\beta=1,2,3}, v_{X_i}\right\rangle}{\left|X_i^{\alpha=1,2,3}-Y_j^{\beta=1,2,3}\right|^3}\Phi(Y_j)w_l\sqrt{\left|J(\xi,\eta)^T\cdot J(\xi,\eta)\right|}. \tag{14}$$

In subsection 3.2, we have successfully transformed the main integral of first order PT by using quadratic element integration. By using Gaussian quadrature as well as the quadratic basis function to accurately approximate the integral over the domain of PT, this method provides a more accurate numerical results and efficient for complex geometrical configuration where the linear elements integration can be no longer efficient. For next section, we further explore the transformation of the main integral using multivariate polynomial interpolation that has been developed by Sukri *et al.,* [11].

3.3 Transformation of Main Integral of First Order PT: Multivariate Polynomial Interpolation

Next, the transformation of the main integral of the first order PT using multivariate polynomial approach is explored for which we try to interpolate the function in the Cauchy Principal Value integral using a polynomial as stated in Eq. (1). In engineering, multivariate polynomial interpolation is a technique that is widely used to approximate the scattered data on a closed surface [19,20]. The flexibility of the polynomial to fit the integral of PT can enhance the precision and the accuracy of PT computation which is crucial for intricate geometries with varying material properties. Hence, by using Eq. (5), let define the function inside the integral to become

$$F(\xi,\eta)=\frac{\left\langle X_i^{\alpha=1,2,3}-Y_j^{\beta=1,2,3},\nu_{X_i}^{\alpha=1,2,3}\right\rangle}{\left|X_i^{\alpha=1,2,3}-Y_j^{\beta=1,2,3}\right|^3}\sqrt{\left|J(\xi,\eta)^T\cdot J(\xi,\eta)\right|}. \tag{15}$$

Function as in (15) can be represented by a polynomial function of ξ and η which is

$$\bar{F}(\xi,\eta)=\sum_{p=0}^{3}\sum_{k+l\geq 0}^{q+l=3}C_{ij}\xi^k\eta^l. \tag{16}$$

Function $\bar{F}(\xi,\eta)$ shall minimize the sum of errors

$$S=\sum_{i=1}^{n}w_i\left[\bar{F}(\xi_i,\eta_i)-F(\xi_i,\eta_i)\right]^2 \tag{17}$$

In order to solve for the polynomial coefficient in (16), partial derivatives are applied to Eq. (17) and all equations will be equated to 0. The coefficient C_{ij} can be easily find by using the matrix system. And by finding the coefficient, the function of F can be interpolate by using $\bar{F}$. By completing the computation similar to quadratic, the matrix for first order PT can be solved. For next section, we presented the result and discussion as these methods are compared for which increasing number of surface elements are used to study the behaviour of the matrix formed by both computations.

4. Results and Discussion

3.1 The Effect of Conductivity towards the Element of First Order PT

In this section, we will discuss the results obtained based on the impact of conductivity changes towards the values of element of first order PT for a sphere with a radius of 0.01, discretized using 3 types of meshing option which is $N=44,72,118$ and $N=620$ meshes. In the context of scattering and imaging problem, conductivity is the critical parameter in determining the behaviour of PT elements, where it influence the electromagnetic field and the object inclusion. Figure 3 until Figure 6 depicts the comparison between the results obtained for first order PT for a spherical object inclusion with different values of conductivity, $k=0.001,0.01,0.1,1.5,10,100,1000,3000,5000,10^4$ and 10^5. Based on the figures (Figure 3 until Figure 6), as the conductivity increases, the values of elements for varying N is diverge from the exact solutions for both multivariate polynomial interpolation as well as quadratic element integration. However, as the number of surface elements increases, the numerical solutions by using multivariate polynomial interpolation shows higher convergence compared to quadratic element integration. This is due to the interpolation function, where, instead of directly compute the integral of PT, the function approximate the results of the integral of K_B^* which lead to reducing values of truncation error. This lead to an accurate numerical solution of first order PT with less computational error.

From the numerical results, it depicts that multivariate polynomial interpolation has higher convergence which suggest that it is more robust for varying conductivity materials rather than quadratic element integration. This is practical in the application of materials characterization and inverse problem for which small errors can significantly affect the results obtained. It also potentially applicable in security screening and archaeological exploration where the truncation error for the results could be minimize. It is also can be important in real time metal detection systems where accurate as well as rapid detection is crucial. However, for higher order polynomial, the computational cost could be higher.

While these results demonstrate the comparative effectiveness of multivariate polynomial interpolation as well as quadratic element integration for computing first order polarization tensor, there's still several limitations that we need to acknowledge. First, relatively simple geometry has been used in this study, whilst most of the application of real life problems involve an irregular and complex shape, which relies on more complex numerical approximations with high level of mesh discretization as well as intricate boundary conditions. Another limitation that can be seen from this study is, the computational cost using multivariate polynomial interpolation which involve high order polynomials is higher, where this can hinder the applicability of the methods to be applied in real life problems.

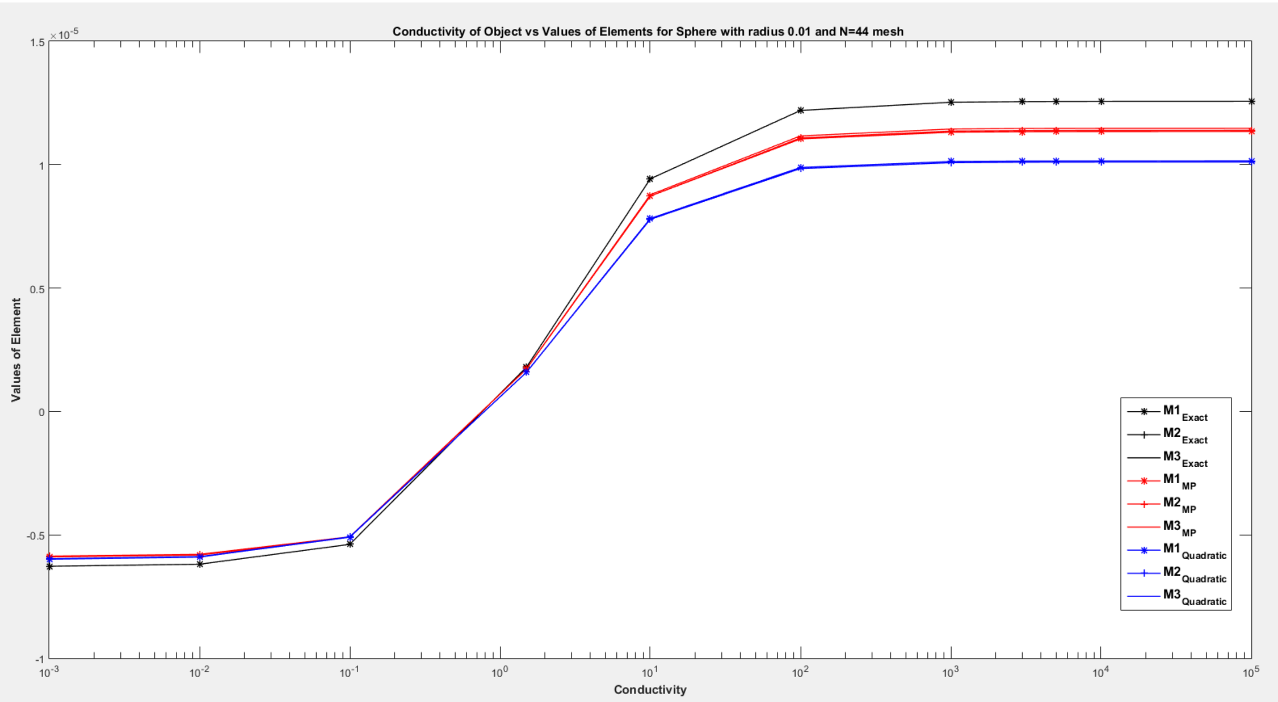

Fig. 3. Element values as a function of conductivity for a sphere with radius 0.01 and $N = 44$ mesh, comparing exact solutions, multivariate polynomial approximations, and quadratic methods for main diagonal M_1, M_2, and M_3

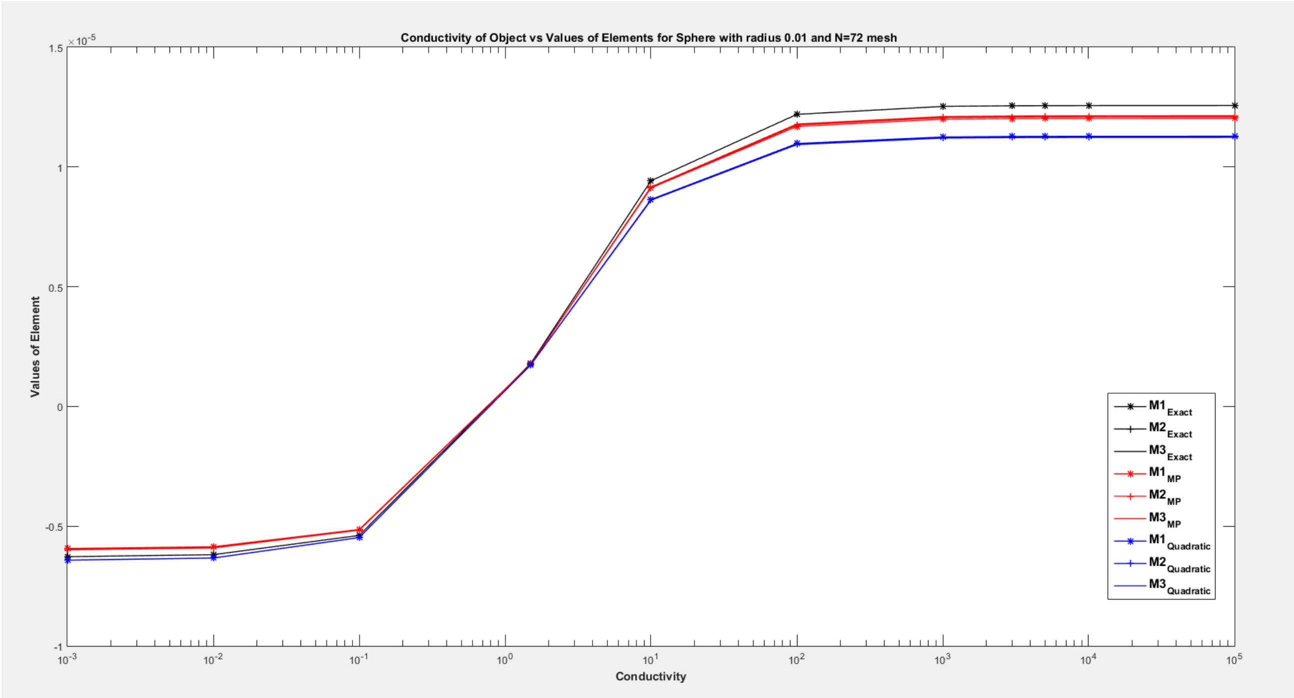

Fig. 4. Element values as a function of conductivity for a sphere with radius 0.01 and $N = 72$ mesh, comparing exact solutions, multivariate polynomial approximations, and quadratic methods for main diagonal M_1, M_2, and M_3

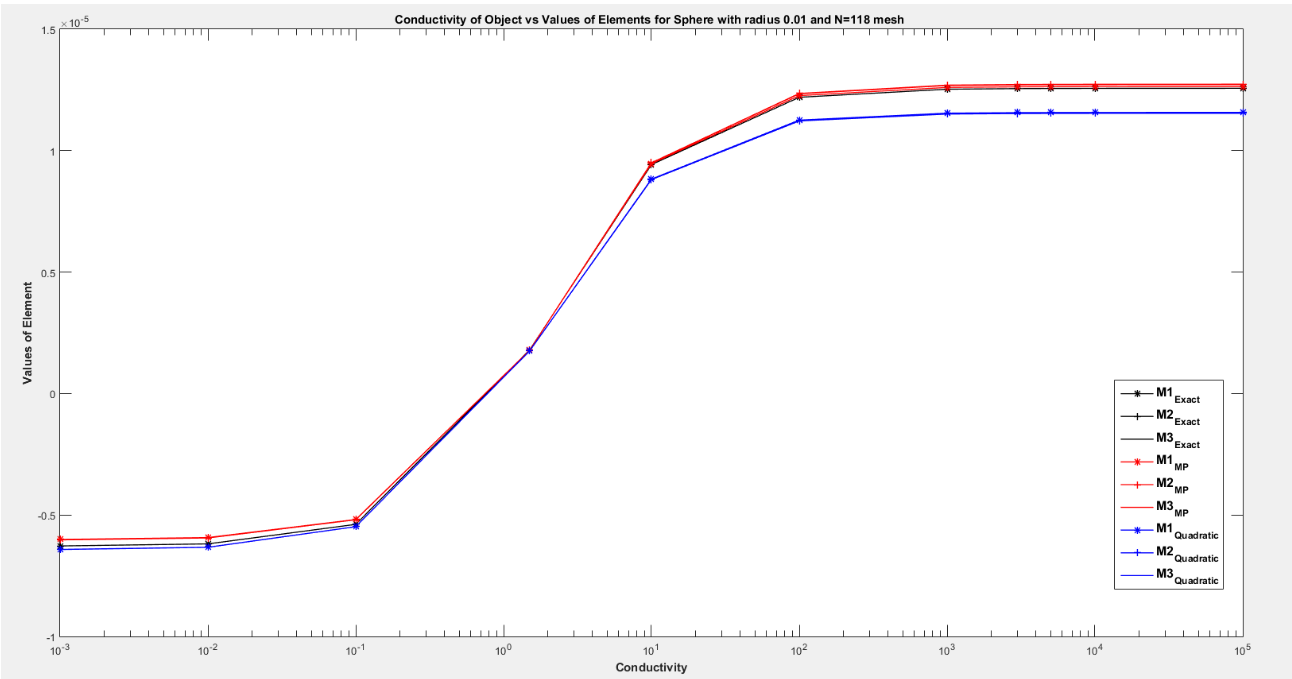

Fig. 5. Element values as a function of conductivity for a sphere with radius 0.01 and $N = 118$ mesh, comparing exact solutions, multivariate polynomial approximations, and quadratic methods for main diagonal M_1, M_2, and M_3

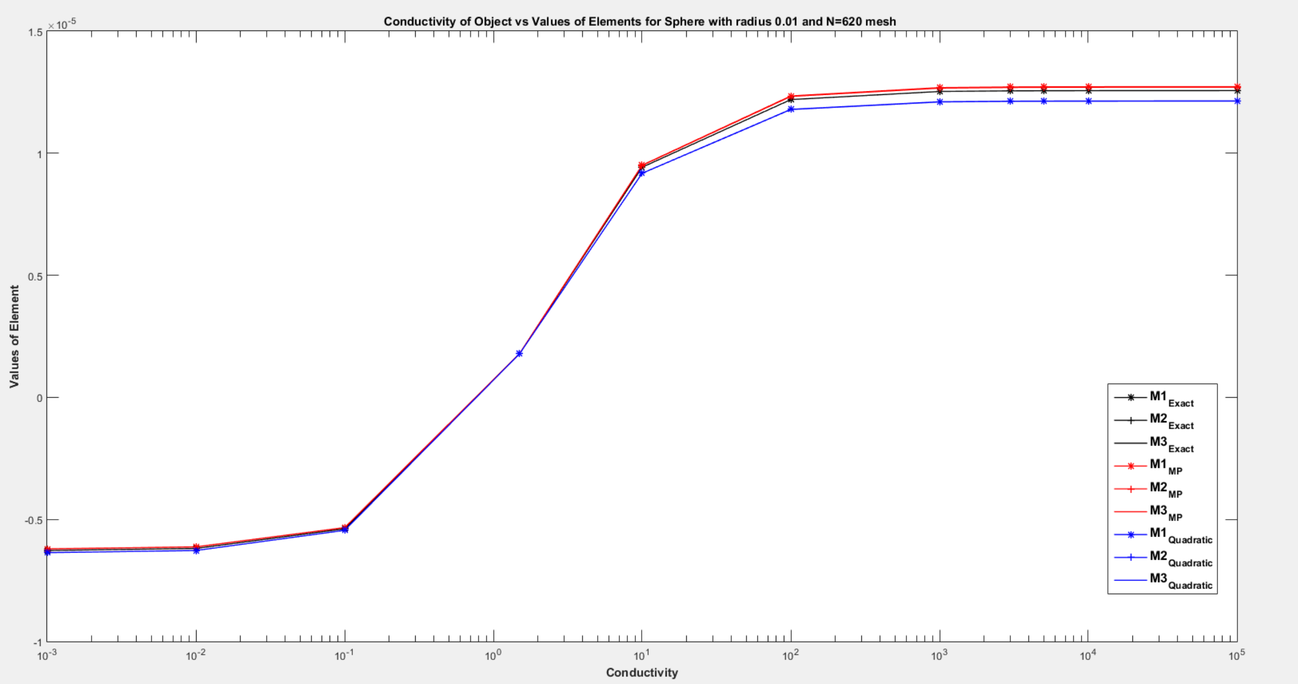

Fig. 6. Element values as a function of conductivity for a sphere with radius 0.01 and $N=118$ mesh, comparing exact solutions, multivariate polynomial approximations, and quadratic methods for main diagonal M_1, M_2, and M_3

5. Conclusions

The comparison between two numerical methods for approximating first-order PT has been analysed and compared for a spherical object inclusion. For this purpose, different discretizations of the mesh, which are $N=44,72,118$ and $N=620$ meshes, were used. This study reveals that when comparing both methods in the context of conductivity, the multivariate polynomial interpolation demonstrates higher convergence compared to quadratic element integration, even as the mesh discretization increases. The truncation error also reduces when multivariate polynomial interpolation is used, leading to higher precision in numerical solutions. This study concludes that multivariate polynomial interpolation is a more effective approach for computing the first-order PT, particularly for objects with varying conductivity, which is widely used in material characterization and electromagnetic modeling. This study could be extended to explore the application of multivariate polynomial interpolation and quadratic element integration for geometrical structures in real-life applications, since most object inclusions are irregular in shape. Besides that, future research can focus on material properties such as the object's permeability and permittivity rather than conductivity. Understanding the influence of the object's properties on the behaviour of PT could be the key starting point to achieving accurate electromagnetic modeling.

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